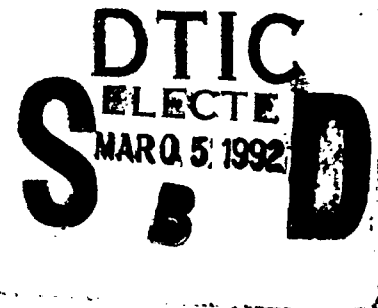


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THESIS

**FREQUENCY, AMPLITUDE, AND PHASE TRACKING OF
NONSINUSOIDAL SIGNAL IN NOISE WITH
EXTENDED KALMAN FILTER**

by

Muhittin Üner

June 1991

Thesis Advisor:

Harold A. Titus

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**Frequency, Amplitude, and Phase Tracking of Nonsinusoidal Signal in Noise
with Extended Kalman Filter**

by

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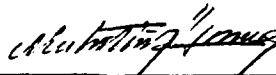
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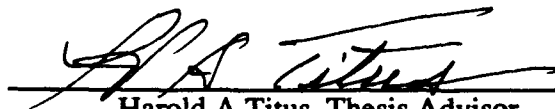
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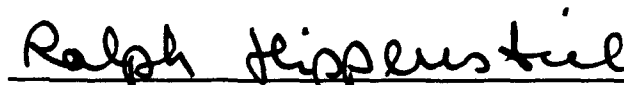


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ABSTRACT

This thesis applies extended Kalman filtering to the problem of estimating frequency, amplitude, and phase of a nonsinusoidal periodic signal contaminated by additive white, Gaussian noise. Parameters will be estimated up to the m th significant harmonic component. It also gives an approach for the case of less than m th significant harmonic components. The estimator will track the signal's fundamental frequency, amplitudes, and phases while these parameters are changing slowly over time. The amplitudes are estimated as if the fundamental frequency estimate is correct; the frequency and the phases of the signal are estimated as if the amplitude estimation is correct. This thesis also contains tracking and the capture behavior of the filter.



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I. INTRODUCTION

The detection, estimation and tracking of signals plays a significant role in many aspects of military and civilian operations. The Kalman filter has been used in tracking problems for many years. Its power comes from the mathematical foundation of statistical optimality. We investigate the behavior of the extended Kalman filter instead of using a linear Kalman filter, as most of the real world problems are non-linear. This thesis studies the detection of non-periodic signals, corrupted by zero mean, white, Gaussian noise using an extended Kalman filter algorithm. The parameters of the signals such as fundamental frequency, harmonic amplitudes and phases, are considered unknown and are estimated by the algorithm.

In nature, most physical signals are approximately harmonic. This problem therefore, has applications in many fields including those of Radar and Sonar. An example of this is in the tracking of propeller driven platforms for submarines and helicopters. The sound of a rotary engine is almost periodic. If the engine speed changes, the sound of rotation changes, resulting in a change of frequency. The amplitudes and phases may also change slowly over time. Knowing the frequency, amplitude and the phase of the signal may allow identification of the class of vessel generating the sound. The frequency of the detected sound tells the speed of the vessel. The amplitude identifies the amount of vibration.

Other applications include the estimation of harmonic signal parameters in the presence of noise to determine a radar's modulated pulse repetition frequency or to investigate noisy biological signals such as heart wave forms.

This thesis is organized into six chapters. Chapters II and III explain the development of the Kalman filter and show the mathematical derivation of the extended Kalman filter algorithm. Chapters IV and V model the physical signal and show the simulation of each model. Chapter VI gives the conclusion of the report. The appendices contain program code applicable to this work.

II. KALMAN FILTER

The object of this chapter is to show the development of the Kalman filter for estimating states from noisy data. The Kalman filter has the desirable quality of maintaining the physical meaning of the system dynamics by utilizing a state space representation. Here the state space model is used as the basic model.

The optimality of the filter holds for systems which are linear and time-invariant (LTI), and corrupted by additive white Gaussian noise (AWGN). This chapter develops the Kalman filter equations for such a system. The development follows closely those given by Lewis, Gelb, and Candy [Ref. 2, 3, 4].

A. DEFINITION OF TERMS

All of the terms used throughout these chapters are defined in Table 2.1 and will be explained as they occur in the derivation.

Terms with a single time subscript (i.e., x_k) refer to the value of the term at that time. Terms with dual subscripts (i.e., $x_{k+1|k}$) refer to the value of the term at the time of the first subscript given observations through the second subscript. The third column of Table 2.1 gives the dimension for each entry.

TABLE 2.1: DEFINITION OF TERMS

System order:	j	
Observation size:	m	
System state:	x_k	$j \times 1$
Transpose of state	x_k^T	$1 \times j$
State transition matrix:	Φ	$j \times j$
State excitation noise:	w_k	$j \times 1$
Observation:	z_k	$m \times 1$
Observation noise:	v_k	$m \times 1$
State estimate:	$\hat{x}_{k+1 k}$	$j \times 1$
Estimate error:	$\tilde{x}_{k+1 k}$	$j \times 1$
Expected value of the error:	$E[\tilde{x}_{k+1 k}]$	$j \times 1$
Error covariance matrix:	$P_{k+1 k+1}$	$j \times j$
Residual:	r_{k+1}	$m \times 1$

B. SYSTEM DYNAMICS

A good model of the system is necessary for estimating parameters of that system. The state space representation of our linear system is given in Equation (2.1), and the measurement process is modeled as in Equation (2.2).

$$x_{k+1} = \Phi x_k + w_k \quad (2.1)$$

$$z_k = Hx_k + v_k \quad (2.2)$$

The physical state of the system (amplitude, frequency, etc.) is described by x and the observed parameters (amplitude, frequency, etc.) are described by observation z . Since the described system is time invariant, both Φ and H are independent of time.

The noise processes are considered as stationary, uncorrelated, zero-mean, additive white Gaussian noise (AWGN) processes. The statistical properties of the noise processes are given below

$$E[w_k] = 0 \quad (2.3)$$

$$E[w_j w_k^T] = Q \delta_{jk} \quad (2.4)$$

$$w_k \sim (0, Q) \quad (2.5)$$

$$E[v_k] = 0 \quad (2.6)$$

$$E[v_j v_k^T] = R \delta_{jk} \quad (2.7)$$

$$v_k \sim (0, R) \quad (2.8)$$

$$E[w_j v_j^T] = 0 \quad (2.9)$$

where δ is the kronecker delta function, defined by:

$$\delta_{jk} = \begin{cases} 1 & \Leftrightarrow j = k \\ 0 & \Leftrightarrow j \neq k \end{cases} \quad (2.10)$$

The matrices Q and R are non-zero, diagonal noise covariance matrices, which denote the power of the noise of the system.

C. RECURSIVE SYSTEM

Before presenting the detail of the Kalman filter equations, we will first introduce the linear and recursive forms of the filter, as shown in Equations (2.11) and (2.12).

$$\hat{x}_{k+1|k} = K_1 \hat{x}_{k|k} \quad (2.11)$$

$$\hat{x}_{k+1|k+1} = K_2 \hat{x}_{k+1|k} + G z_{k+1} \quad (2.12)$$

K_1 is a system matrix, K_2 and G are the time-varying weighting matrices. The current estimate is a combination of the previous estimate and the current observation.

D. DISCRETE KALMAN FILTER

The Kalman filter may be derived by optimizing the assumed form of the linear estimator. An optimal system is generally considered to be any system which either minimizes a cost function or optimizes a performance function.

This experiment required the establishment of a filter that gives an unbiased estimate and minimizes error. An unbiased estimate has an expected error value of zero. The Constants K_1 in Equation (2.11) and K_2 in Equation (2.12) are chosen to make the estimate unbiased. The constant G is chosen to minimize a cost function of the expected error.

Following measurement, an equation for the estimation error can be obtained from Equation (2.12) by substitution of the measurement Equation (2.2) and the defining the relations as (tilde denotes estimation error):

$$\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} \quad (2.13)$$

$$\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1} \quad (2.14)$$

The expected values satisfy

$$E[\tilde{x}_{k+1|k+1}] = E[\tilde{x}_{k+1|k}] = 0 \quad (2.15)$$

and

$$\tilde{x}_{k+1|k+1} = [K_2 + GH - I]x_{k+1|k} + K_2x_{k+1|k} + K_2v_k. \quad (2.16)$$

By taking the expected value of both sides and letting $E[\tilde{x}_{k+1|k}] = 0$, $E[v_k] = 0$, and $E[x_{k+1|k+1}] = 0$ (i.e. unbiased estimator), the term in square brackets in Equation (2.16) is required to be zero.

$$K_2 = I - GH \quad (2.17)$$

Combining Equation (2.14) and (2.17) gives the estimator the form of

$$\hat{x}_{k+1|k+1} = [I - GH]\hat{x}_{k+1|k} + Gz_{k+1} \quad (2.18)$$

or, alternatively

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + G[z_{k+1} - H\hat{x}_{k+1|k}]. \quad (2.19)$$

Following the same procedure, the value of K_1 can be determined by substituting Equations (2.1) and (2.11) into Equation (2.13) as below:

$$\tilde{x}_{k+1|k} = \Phi x_k + w_k - K_1\hat{x}_{k|k} - \Phi\hat{x}_{k|k}. \quad (2.20)$$

By taking the expected value of both sides and applying Equations (2.3) and (2.15), we get

$$K_1 = \Phi \quad (2.21)$$

so that Equation (2.11) can be written in the form

$$\hat{x}_{k+1|k} = \Phi \hat{x}_{k|k}. \quad (2.22)$$

Equation (2.22) is known as the time update portion of the algorithm and gives the prediction $\hat{x}_{k+1|k}$ of the state at time $k+1$, along with the associated error covariance $P_{k+1|k}$. Also, Equation (2.19) is known as measurement update portion which provides a correction based on the measurement z_{k+1} at time $k+1$ to yield the net *a posteriori* estimate of x_{k+1} and its error covariance of $P_{k+1|k+1}$. This is known as predictor-corrector formulation of the discrete Kalman filter.

E. ERROR COVARIANCE UPDATE

To obtain a measure of confidence for the estimate, we need to find the error covariance. The error covariance matrices are defined by Equations (2.23) and (2.24).

$$P_{k+1|k} = E[\tilde{x}_{k+1|k} \tilde{x}_{k+1|k}^T] \quad (2.23)$$

$$P_{k+1|k+1} = E[\tilde{x}_{k+1|k+1} \tilde{x}_{k+1|k+1}^T] \quad (2.24)$$

The corresponding estimation error is of the form

$$\tilde{x}_{k+1|k+1} = [I - GH] \tilde{x}_{k+1|k} + Gv_k. \quad (2.25)$$

Rewriting Equation (2.24) as

$$P_{k+1|k+1} = E \{ (I - GH) \tilde{x}_{k+1|k} [\tilde{x}_{k+1|k}^T (I - GH)^T + v_k^T G^T] + G v_k [\tilde{x}_{k+1|k}^T (I - GH)^T + v_k^T G^T] \} \quad (2.26)$$

and using Equations (2.23), (2.7), and the result of uncorrelated measurement errors we have,

$$E [\tilde{x}_{k+1|k} v_k^T] = E [v_k \tilde{x}_{k+1|k}^T] = 0, \quad (2.27)$$

and

$$P_{k+1|k+1} = (I - GH) P_{k+1|k} (I - GH)^T + GRG^T. \quad (2.28)$$

The error covariance matrix gives the expected magnitude of the estimation error.

F. OPTIMUM CHOICE OF KALMAN GAIN

The criterion for choosing the gain is to minimize a weighted scalar sum of the diagonal elements of the error covariance matrix. We will choose the value of G which satisfies Equation (2.29).

$$G: \min_G \{J_{k+1}\} = E [\tilde{x}_{k+1|k+1} \tilde{x}_{k+1|k+1}^T] \quad (2.29)$$

Equation (2.29) can be written in terms of the error covariance as

$$G: \min_G \{J_{k+1}\} = \text{trace} (P_{k+1|k+1}). \quad (2.30)$$

This is equivalent to minimizing the length of the estimation error vector. To find the value of the gain which provides a minimum, it is necessary to take the partial derivative

of J_{k+1} with respect to G and set it to zero. Taking the partial derivative of J_{k+1} requires the use of the matrices A and B , where B is symmetric.

$$\frac{\partial}{\partial A} \text{trace}(ABA^T) = 2AB \quad (2.31)$$

Inserting Equation (2.28) into Equation (2.30) and applying Equation (2.31) gives the value

$$-2(I - GH)P_{k+1|k}H^T + 2GR = 0. \quad (2.32)$$

Solving for G , we have

$$G = P_{k+1|k}H^T [HP_{k+1|k}H^T + R]^{-1} \quad (2.33)$$

which is referred to as the Kalman gain matrix.

G. KALMAN FILTER EQUATIONS

The set of recursive equations in Table 2.2 provide the time varying optimal gain matrix and the error analysis of the estimate.

TABLE 2.2: KALMAN FILTER EQUATIONS

Time update (effect of system dynamics)

Error covariance:

$$P_{k+1|k} = \Phi P_{k|k} \Phi^T + Q \quad (2.34)$$

Estimate:

$$\hat{x}_{k+1|k} = \Phi \hat{x}_{k|k} \quad (2.35)$$

Measurement update (effect of measurement z)

Error covariance:

$$P_{k+1|k} = [(P_{k+1|k})^{-1} + H^T R^{-1} H]^{-1} \quad (2.36)$$

Residual:

$$e_{k+1} = z_{k+1} - H \hat{x}_{k+1|k} \quad (2.37)$$

Estimate:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + P_{k+1|k} H^T R^{-1} (z_{k+1} - H \hat{x}_{k+1|k}) \quad (2.38)$$

Alternative Measurement Update Equations:

$$G = P_{k+1|k} H^T (H P_{k+1|k} H^T + R)^{-1} \quad (2.39)$$

$$P_{k+1|k} = (I - GH) P_{k+1|k} \quad (2.40)$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + G (z - H \hat{x}_{k+1|k}) \quad (2.41)$$

These equations require the following initial conditions, $\hat{x}_{0|0}$ and $P_{0|0}$.

There are many equivalent formulations for Equations (2-33) and (2-35). Using the matrix inversion lemma the latter can be written as

$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} H^T [H P_{k+1|k} H^T + R]^{-1} H P_{k+1|k}, \quad (2.42)$$

Note, Equation (2.42) requires only a $p \times p$ matrix inversion. Equation (2.36) requires the inversion of two $v \times v$ matrices, and the number of measurements for p is usually less than for v . If $|P_{k+1|k}| = 0$, then Equation (2.44) must be used.

If it is often convenient to replace the measurement update equations by alternative equations such as those in Table 2.1.

The Kalman filter gain is the weighting that determines the influence of the residual in updating the estimate. It is important to remember that the equations of Table 2.2 are only optimal when the system is linear, and when the *a priori* knowledge of the noise is available.

III. EXTENDED KALMAN FILTER

In this section, an approximate solution to the non-linear filtering problem defined below is developed. This solution involves the linearization of a non-linear process traversing about a reference trajectory and the modification or extension of the linear Kalman filter algorithm using the linearized model.

In practice, many processes are non-linear rather than linear. Coupling non-linearities with noisy data makes the signal-processing problem a challenging one. Instead of extending this solution to the continuous case, the extended Kalman filter algorithm was developed to test for the discrete non-linear system.[Ref. 2] and [Ref. 3].

A. SYSTEM DYNAMICS

Let us discuss the non-linear system of the form

$$x_{k+1} = \Phi x_k + w_k \quad (3.1)$$

$$z_{k+1} = h(x_{k+1}) + v_{k+1}, \quad (3.2)$$

where $w_k \sim (0, Q)$ and $v_k \sim (0, R)$ are white noise processes uncorrelated with each other.

B. APPROXIMATE MEASUREMENT UPDATE

In order to find a measurement update that can be conveniently programmed, $h(x_{k+1})$ is expanded into a Taylor series about $\hat{x}_{k+1|k}$, an *a priori* estimate at time $k+1$.

$$h(x_{k+1}) = h(\hat{x}_{k+1|k}) + \left. \frac{\partial h}{\partial x} \right|_{x=\hat{x}_{k+1|k}} (x_{k+1} - \hat{x}_{k+1|k}) + \text{higher order terms} \quad (3.3)$$

Neglecting the higher order terms (H.O.T.)

$$h(x_{k+1}) = h(\hat{x}_{k+1|k}) + \left. \frac{\partial h}{\partial x} \right|_{x=\hat{x}_{k+1|k}} \tilde{x}_{k+1|k} \quad (3.4)$$

representing the Jacobian as;

$$H(x, k) = \left. \frac{\partial h}{\partial x} \right|_{x=x, k} \quad (3.5)$$

using Equations (3.4) and (3.5) can be used to proceed with the Kalman filter analysis.

C. APPROXIMATE TIME UPDATE

To obtain a complete filtering algorithm, update equations that account for measurement data are needed. We choose a linear, recursive set of equations for the estimate,

$$\hat{x}_{k+1|k+1} = a_{k+1|k} + G_{k+1} z_{k+1}, \quad (3.6)$$

where the vector a_k and the gain matrix G_k are to be determined. Proceeding with arguments similar to those used in Chapter II, we define the estimation error as

$$\tilde{x}_{k+1|k+1} \equiv \hat{x}_{k+1|k+1} - x_{k+1} \quad (3.7)$$

$$\tilde{x}_{k+1|k+1} \equiv \hat{x}_{k+1|k} - x_{k+1}. \quad (3.8)$$

Equations (3.6) and (3.7) and (3.8) are combined with Equation (3.2) to produce the following expression for the estimation error:

$$\bar{x}_{k+1|k+1} = \Delta a_{k+1} + G_{k+1}h(x_{k+1}) + G_{k+1}v_{k+1} + \bar{x}_{k+1|k} - \hat{x}_{k+1|k} \quad (3.9)$$

One required condition is that the estimate is unbiased. Applying this requirement to Equation (3.9) and letting $E[x_{k+1|k}] = E[v_k] = 0$, we obtain

$$a_{k+1} + G_{k+1}h(\hat{x}_{k+1|k}) - \hat{x}_{k+1|k} = 0. \quad (3.10)$$

By defining the residual as the difference between the observation and the expected value of the observation in Equation 3.11

$$e_{k+1} = z_{k+1} - h(\hat{x}_{k+1|k}), \quad (3.11)$$

solving Equation (3.10) for a_{k+1} , and substituting the result into Equation (3.6), and combining it with Equation (3.11) yields the extended Kalman filter estimate equation

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + G_{k+1}r_{k+1}. \quad (3.12)$$

D. ESTIMATION ERROR COVARIANCE

Using the definition of error covariance

$$P_{k+1|k} = E[\bar{x}_{k+1|k}\bar{x}_{k+1|k}^T] \quad (3.13)$$

and

$$P_{k+1|k+1} = E[\tilde{x}_{k+1|k+1}\tilde{x}_{k+1|k+1}^T] \quad (3.14)$$

the error-update equations may be derived. Taking Equation (3.10) into account, Equation (3.9) becomes

$$\tilde{x}_{k+1|k+1} = \tilde{x}_{k+1|k} + G_{k+1}[h(x_{k+1}) - h(\hat{x}_{k+1})] + G_{k+1}v_{k+1}. \quad (3.15)$$

Using this, to generate the error covariance P_{k+1} in terms of the yet undetermined G_{k+1} we can write

$$\begin{aligned} P_{k+1|k+1} &= P_{k+1|k} + G_{k+1}E\{[h(x_{k+1}) - h(\hat{x}_{k+1})][h(x_{k+1}) - h(\hat{x}_{k+1})]^T\}G_{k+1}^T + \\ &E\{\tilde{x}_{k+1|k}[h(x_{k+1}) - h(\hat{x}_{k+1})]^T\}G_{k+1}^T + \\ &G_{k+1}E\{[h(x_{k+1}) - h(\hat{x}_{k+1})]\tilde{x}_{k+1|k}^T\} + G_{k+1}R_{k+1}G_{k+1}^T, \end{aligned} \quad (3.16)$$

where

$$R_k = E[v_k v_k^T].$$

The gain G_{k+1} is now selected to minimize P_{k+1} . Differentiating P_{k+1} with respect to G_{k+1} and solving for G_{k+1} results in the desired optimal gain matrix.

$$G_{k+1} = E \{ (\tilde{x}_{k+1|k}) [h(x_{k+1}) - h(\hat{x}_{k+1})]^T \} \times \\ \{ E \{ [h(x_{k+1}) - h(\hat{x}_{k+1})] [h(x_{k+1}) - h(\hat{x}_{k+1})]^T \} + R_{k+1} \}^{-1}. \quad (3.17)$$

Substituting this into Equation (3.16) and simplifying yields

$$P_{k+1|k+1} = P_{k+1|k} + G_{k+1} E \{ [h(x_{k+1}) - h(\hat{x}_{k+1})] \tilde{x}_{k+1|k}^T \}. \quad (3.18)$$

The complete linear estimate update due to non-linear measurement is given by Equations (3.15), (3.18) and (3.19). However, these equations are impractical to implement because they depend on conditional moments of x_{k+1} in computing $h(\hat{x}_{k+1})$. In order to simplify the computation, we will expand $h(x_{k+1|k+1})$ into a power series about $\hat{x}_{k+1|k}$ as follows:

$$h(x_{k+1|k+1}) = h(\hat{x}_{k+1|k}) + H(\hat{x}_{k+1|k})(x_{k+1|k+1} - \hat{x}_{k+1|k}) + \dots \quad (3.19)$$

where

$$H(\hat{x}_{k+1|k}) = \left. \frac{\partial}{\partial x} h(x) \right|_{x=\hat{x}_{k+1|k}}.$$

Truncating the above series after the first two terms, substituting the resulting approximation into Equations (3.18), (3.19), and carrying out the indicated expectation operations results in a measurement error covariance update as follows:

$$P_{k+1|k+1} = [I - G_{k+1} H(\hat{x}_{k+1|k})] P_{k+1|k}. \quad (3.20)$$

This equation represents an approximate, linear measurement update for the error covariance. The residual is computed using the non-linear measurement function $h(x)$ evaluated about the *a priori* estimate $\hat{x}_{k+1|k}$. The error covariance is found using the Jacobian matrix.

TABLE 3.1: EXTENDED KALMAN FILTER EQUATIONS

System model and measurement model:

$$\dot{x} = f(x, t) + w(t) \quad (3.21)$$

$$z_k = h(x(k)) + v_k \quad (3.22)$$

$$w(t) \sim (0, Q), v_k \sim (0, R) \quad (3.23)$$

Time update:

Estimate (state prediction):

$$\hat{x}_{k+1|k} = f\hat{x}_{k|k} \quad (3.24)$$

Jacobians:

$$\Phi(x, t) = \frac{\partial f(x, t)}{\partial x} \quad (3.25)$$

$$H(x) = \frac{\partial h(x, k)}{\partial x} \quad (3.26)$$

Error covariance (covariance prediction):

$$P_{k+1|k} = \Phi \hat{x}_{k+1|k} P_{k|k} \Phi^T + Q \quad (3.27)$$

Measurement update:

Kalman gain:

$$G_{k+1} = P_{k+1|k} H^T(\hat{x}_{k+1|k}) [H(\hat{x}_{k+1|k}) P_{k+1|k} H^T(\hat{x}_{k+1|k}) + R]^{-1} \quad (3.28)$$

Error covariance (covariance correction):

$$P_{k+1|k+1} = [I - G_{k+1} H(\hat{x}_{k+1|k})] P_{k+1|k} \quad (3.29)$$

Estimate (state correction):

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + G_{k+1} [z_{k+1} - h(\hat{x}_{k+1|k})] \quad (3.30)$$

The covariance and gain equations are identical to those in Table 2.2, but with the Jacobian values A and H linearized about $\hat{x}_{k+1|k}$. Note that P and G are now functions of the current state estimate, which is a conditional mean and therefore a single realization of a stochastic process. It also should be realized that the measurement times do not need to be equally spaced. The time update is performed over any interval during which no data are available. When data become available, a measurement update is performed. This means that an indication of missing measurements can be obtained, and pure prediction in the absence of data can easily be achieved by using the extended Kalman filter.

Higher-order approximations to the optimal non-linear updates can also be derived by retaining higher-order terms in the Taylor series expansions.

IV. POLAR MODEL OF THE SYSTEM

Consider an approximately periodic, non-sinusoidal signal, in additive white Gaussian noise. A non-sinusoidal signal may be considered to consist of an infinite number of sinusoidal components. Three sets of parameters can characterize the signal: the fundamental frequency, the amplitude of each harmonic component, and the phase of each harmonic component.

The signal is not exactly periodic since frequencies, amplitudes and phases change slowly over time. We will keep the same notation as defined in Nehorai and Porat [Ref. 6] and Parker and Anderson [Ref. 7]. For this purpose, we will first take a periodic signal $y(t)$ with a zero d.c. component. A Fourier series representation of this signal can be written as:

$$y(t) = \sum_{k=1}^{\infty} r_k \sin(k\omega_f t + \phi_k) \quad (4.1)$$

In this paper a discrete time domain (i.e. $t = 0, 1, 2, \dots$) rather than a continuous domain will be used. As our signal $y(t)$ is not exactly periodic, but has a slowly time varying frequency ω_f , amplitudes r_k , and phases ϕ_k , we can state

$$\omega_f = \omega_f(t) \quad (4.2)$$

$$r = r_k(t) \quad (4.3)$$

$$\phi_k = \phi_k(t) \quad (4.4)$$

where $w_f(t)$, $r_k(t)$ and $\phi_k(t)$ are nearly constant over a few periods of $y(t)$. In other words;

$$\frac{|w_f(t+1) - w_f(t)|}{w_f(t)} \equiv 0 \quad (4.5)$$

$$\frac{|r_k(t+1) - r_k(t)|}{w_f(t)} \equiv 0 \quad (4.6)$$

$$\frac{|\phi_k(t+1) - \phi_k(t)|}{w_f(t)} \equiv 0 \quad (4.7)$$

In the model of $y(t)$, the quantities $w_f(t)$, $r_k(t)$ and $\phi_k(t)$ are the instantaneous frequency, amplitudes, and phases of the signal. We will call the model given in Equation (4.1) the polar model and the model given in Chapter V the rectangular model [Ref.7].

We assume for both models that the signal $y(t)$ is corrupted by noise. The measurements are given by

$$z(t) = y(t) + v(t). \quad (4.8)$$

The task is to estimate the values $r_1(t), \dots, r_m(t), w_f(t), \phi_1(t), \dots, \phi_m(t)$ from the measurements, where m denotes the number of the significant harmonics. Parameters are only estimated up to m th harmonics. The higher harmonics are assumed to be negligible. A total of $2m+1$ parameters must be estimated.

As explained in Parker and Anderson [Ref.7], we are also estimating amplitudes as well as the fundamental frequency. This is required to establish an estimator that uses both the energy in the fundamental and the energy in the higher harmonics to estimate the frequency of the signal. The information about the frequency contained in any harmonic de-

depends on the energy in that harmonic. If a particular harmonic component is strong, then the estimator of the frequency components should give more weight to the information available in the strong harmonic and less weight to the information available in the weak harmonics.

Estimation of the harmonic amplitude also assists in estimating the frequency. The estimator determines the frequency by first estimating the harmonic amplitudes. Knowledge of the frequency and the phases in the polar model also assists in the calculation of the harmonic amplitudes.

A. THE ESTIMATOR

Given a noisy measurement sequence and the associated models as shown in Fig. 4.1[Ref.4], the Kalman filter can be thought of as an estimator that produces three types of outputs. The first filter can be named as a state estimator or reconstructor. It reconstructs estimates of the state $x(t)$ from noisy measurements $y(t)$. This kind of model may be thought of as the means to implicitly extract $x(t)$ from $y(t)$.

The second filter, may be thought of as a measurement filter that accepts noisy sequences of input and produces a filtered sequence of output. Finally, the estimator can be thought as a whitening filter that accepts noisy correlated measurements and produces uncorrelated or white residues $e(t) = z(t) - \hat{z}(t|t-1)$. For our purposes, we will concentrate on the measurement filter form.

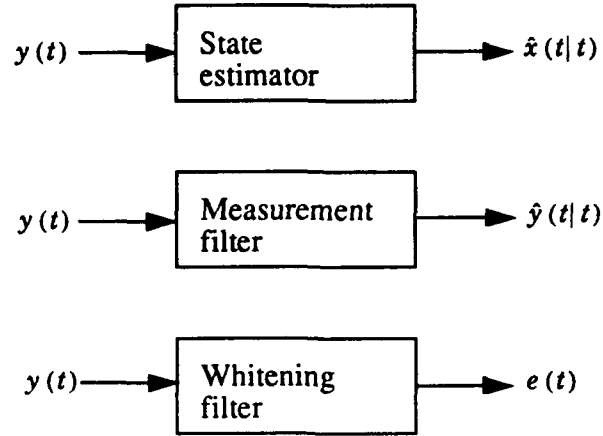


Figure 4.1 Various representations of the Kalman filter

As mentioned before, the extended Kalman filter will be used to estimate frequency, phases, and amplitudes of an almost periodic signal imbedded in noise. First, we need a state space representation of the signal defined in Equation (4.1). Therefore,

$$x(t+1) = \Phi x(t) + w(t) \quad (4.9)$$

$$z(t) = y(t) + v(t) \quad (4.10)$$

$$z(t) = h(x(t)) + v(t) \quad (4.11)$$

and

$$x(t) = [r_1(t), r_2(t), \dots, r_m(t), w_f(t), \phi_1(t), \phi_2(t), \dots, \phi_m(t)]^T \quad (4.12)$$

and

$$\Phi = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_m \end{bmatrix} \quad (4.13)$$

where, I_m is a m th order identity matrix,

$$h(x(t)) = \sum_{k=1}^m r_k(t) \sin(k\omega_f t + \phi_k) \quad (4.14)$$

and $w(t)$ is white Gaussian noise, with a zero mean and a variance

$$E[w(t)w^T(t)] = Q. \quad (4.15)$$

The observation noise $v(t)$ is also white Gaussian noise, with zero mean and has a variance

$$E[v(t)v^T(t)] = R \quad (4.16)$$

$$(4.17)$$

and is uncorrelated with $w(t)$.

$$E[w(t)v(t)] = 0 \quad (4.18)$$

We will have a Q matrix which is diagonal. Its value will be given in the simulation section. From Equation (4.9), it can be concluded that the harmonic amplitudes evolve randomly over time. Also, the same argument is true for $\omega_f(t)$, the fundamental frequency, and the $\phi_k(t)$ phases of the signal. The rate of the random walk will be determined by the

diagonal Q matrix. A zero Q matrix will correspond to constant amplitude, frequency and phase. In order to estimate $\hat{x}(t|t)$ or $\hat{x}(t|t-1)$ of $x(t)$ from the measurement $z(t)$, the extended Kalman filter will be applied. Here $\hat{x}(t|t)$ denotes the estimation of $x(t)$, given measurements $\{z(\tau) | \tau = 0, 1, 2, \dots, t\}$ up to and including time t . The value $\hat{x}(t|t-1)$ is an estimate of $x(t)$, given $z(\tau)$ up to time $t-1$.

$$\hat{x}(t|t) = \hat{x}(t|t-1) + G(t) [z(t) - h(\hat{x}(t|t-1))] \quad (4.19)$$

$$\hat{x}(t+1|t) = \Phi \hat{x}(t|t) \quad (4.20)$$

$$G(t) = P(t) H^T(t) (H(t) P(t) H^T(t) + R)^{-1} \quad (4.21)$$

$$P(t+1) = \Phi [P(t) - G(t) H(t) P(t)] \Phi^T + Q \quad (4.22)$$

where $H(t)$ is the Jacobian of $h(t)$. That is:

$$H(t) = \frac{\partial h(\hat{x}(t|t-1))}{\partial \hat{x}(t|t-1)}$$

$$H(t) = \begin{bmatrix} \sin(\hat{\omega}_f(t|t-1)t + \hat{\phi}_1(t|t-1)) \\ \sin(\hat{\omega}_f(t|t-1)t + \hat{\phi}_2(t|t-1)) \\ \dots \\ \sin(\hat{\omega}_f(t|t-1)t + \hat{\phi}_m(t|t-1)) \\ \Im(t|t-1) \\ \hat{r}_1(t|t-1) \cos(\hat{\omega}_f(t|t-1)t + \hat{\phi}_1(t|t-1)) \\ \hat{r}_2(t|t-1) \cos(\hat{\omega}_f(t|t-1)t + \hat{\phi}_2(t|t-1)) \\ \dots \\ \hat{r}_m(t|t-1) \cos(\hat{\omega}_f(t|t-1)t + \hat{\phi}_3(t|t-1)) \end{bmatrix}^T \quad (4.23)$$

where,

$$\mathfrak{I}(t|t-1) = \sum_{k=1}^4 \hat{r}_k t \cos(\hat{w}_f(t|t-1)t + \hat{\phi}_k(t|t-1))$$

and the initial values are

$$\hat{x}(0) = E[x(0)] = \bar{x}(0)$$

and

$$P(0) = E[(x(0) - \bar{x}(0))(x(0) - \bar{x}(0))^T].$$

B. SIMULATIONS

In the simulation we let

$$\begin{aligned} z(t) = & r_1 \sin(2\pi 0.04t + 0.01) + r_2 \sin(4\pi 0.04t + 0.01) + r_3 \sin(6\pi 0.04t + 0.01) + \\ & r_4(8\pi 0.04t + 0.01) + v(t) \end{aligned} \quad (4.24)$$

where $v(t)$ is a zero mean, white Gaussian noise process with a variance of 0.1. The fundamental frequency of the signal $w_f = 2\pi 0.04$ and the amplitudes of the signal r_k are constant over time. We use the amplitudes as written below.

$$r_k = \frac{r_{k-1}}{2} \quad k = 2, 3, 4, \quad (4.25)$$

where r_1 is chosen for a desired signal to noise ratio as given by

$$SNR = 10 \log \left(\frac{\left(\sum_{k=1}^4 \frac{1}{2} r_k^2 \right)}{0.1} \right) \quad (4.26)$$

For the simulations, the filter program was written in MATLAB (See Appendix A) with the following initial conditions:

$$P_1(0) = 2.4 \quad (4.27)$$

$$P_2(0) = 0 \quad (4.28)$$

$$P_3(0) = 0 \quad (4.29)$$

$$P_4(0) = 0 \quad (4.30)$$

$$\dot{w}_f(0) = 2\pi 0.03 \quad (4.31)$$

$$\hat{\phi}_1(0) = 0 \quad (4.32)$$

$$\hat{\phi}_2(0) = 0 \quad (4.33)$$

$$\hat{\phi}_3(0) = 0 \quad (4.34)$$

$$\hat{\phi}_4(0) = 0 \quad (4.35)$$

$$P(0) = \text{diag}\{6, 5, 3, 2, 0.08, 0.01, 0.01, 0.01, 0.01\} \quad (4.36)$$

As stated in Candy [Ref. 4], the variation of the gain is related to the variations in Q and P . Uncertainty in the model can be characterized by the process noise covariance. For a large Q , P is large, which indicates a high uncertainty or an inadequate model. small Q , P is small, indicating an adequate model. Therefore, the Kalman gain can be thought of as a ratio of process to measurement noise; that is,

$$G \propto \frac{Q}{R} \quad (4.37)$$

It can be seen that the variations in G are proportional to the variations of Q . The Kalman estimator can be perceived as a deterministic filter with a time-varying bandwidth as determined by the filter gain. As Q increases, both G and the filter bandwidth increase. Thus, the filter transient performance will be faster. The same effect can be observed by a small R . Conversely, if Q decreases, G decreases, which decreases the bandwidth of the filter, and hence, the filter transient response is slower.

The steady-state value of the error covariance matrix P increases (decreases) with a corresponding increasing (decreasing) value of the Q matrix. We conclude that P is also proportional to Q . We also note a similar effect by varying R .

We investigated the effect of changing the initial conditions of P on the algorithm performance. As long as the filter is stable and the state-space model is completely controllable and observable, then

$$\lim_{t \rightarrow \infty} P(t) = P(\text{small}). \quad (4.38)$$

Thus, an estimator error approaching zero implies that the state estimate converges to a true value, given that enough data exists. The initial estimates will affect the transient performance of the algorithm, since a large initial $P(0)$ gives a large $G(0)$, and therefore heavily weights the initial measurements and ignores the model.

In the program, we used $R = 0.1$, while

$$Q = \text{diag} \{q_1, q_1, q_1, q_1, q_2, q_2, q_2, q_2, q_2\} \quad (4.39)$$

where

$$q_1 = 1 \times 10^{-3} \text{ and } q_2 = 1 \times 10^{-7}.$$

The entries of Q and R roughly determine the bandwidth of the estimator. This also affects the capture properties, tracking properties, and steady-state error. The capture property can also be influenced by the choice of $P(0)$. The values used in this work were largely obtained by trial and error.

Figure 4.6 shows the true and estimated values of the fundamental frequency from 450 samples. The estimated value of the frequency tracks the true value after $t = 30$ and locks onto it very closely. Figure 4.2 shows the true and the estimated values of the amplitude of the first harmonic component. Tracking is achieved after $t = 50$. Figure 4.3 shows the estimated and the true amplitudes of the second harmonic component. Tracking is achieved after $t = 50$, although the initial estimate value $\hat{r}_2(0)$ is zero. The other harmonic amplitudes and their estimates show similar responses, as in Figures 4.4 and 4.5.

The phase error of the fundamental component is shown in Figure 4.7. The error is close to zero after $t = 350$. A similar response can be seen in Figure 4.8 for the phase error of the second harmonic. Figure 4.9 shows the phase error of the third harmonic. At $t = 450$, the error is 0.00025. Figure 4.10 shows the phase error of the fourth harmonic, which has a steady state error of 0.007 after $t = 250$. The phase error of the second and the higher harmonics is higher than in the first harmonic. This is probably because the initial estimate of the second and the higher harmonic amplitudes were zero.

Since higher harmonics are assumed to be negligible we only considered tracking signals with less than four harmonics. For one case, we accepted a signal consisting of two harmonics. For the simulations, we let

$$z(t) = r_1 \sin(2\pi 0.04t + 0.01) + r_3 \sin(6\pi 0.04t + 0.01) + v(t) \quad (4.42)$$

and the second and the fourth harmonics are assumed to be missing. The filter could not initially track the initial error covariance matrix used with four harmonics. So we rearranged the initial value of the error covariance matrix as

$$P(0) = \text{diag}\{2, 1, 0.5, 0.2, 0.04, 0.01, 0.01, 0.01, 0.01\}. \quad (4.43)$$

We used the MATLAB program in Appendix B for the simulations. The results are given in Figures 4.11 to Figure 4.15. Figures 4.11 and 4.12 show the true and estimated values of the first and the third harmonics, respectively. For both cases tracking is achieved after $t = 30$. The second and the fourth harmonic true and estimated values were around zero. These graphs are not included. Figure 4.13 shows the frequency tracking of the signal. Its transient response was similar to the one obtained for the case of four harmonics. Figures 4.14 and 4.15 show the phase error of the first and the third harmonics, respectively. The phase error of the third harmonic was worse than the first harmonic because of the poor initial value of the third harmonic.

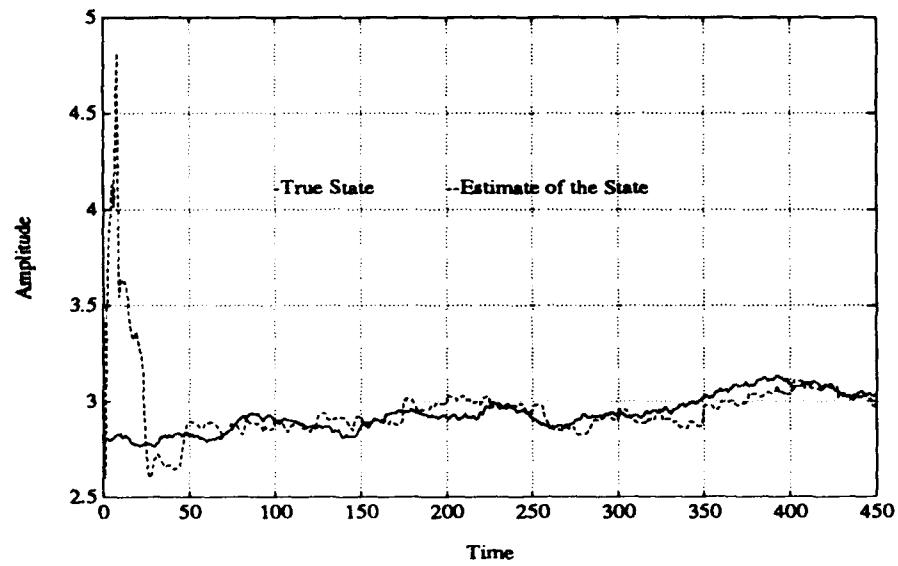


Figure 4. 2 True and the estimate value of the amplitude of the first harmonic

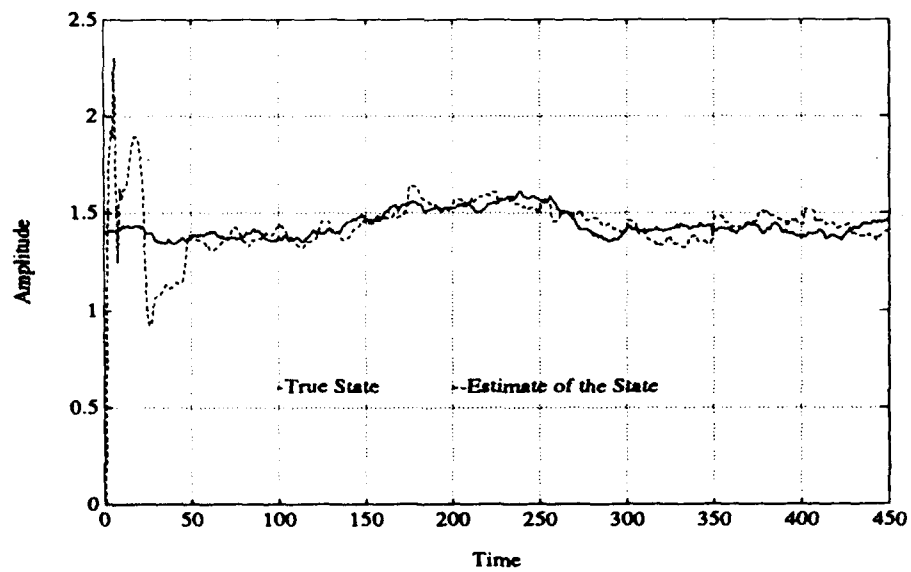


Figure 4. 3 True and the estimate value of the amplitude of the second harmonic

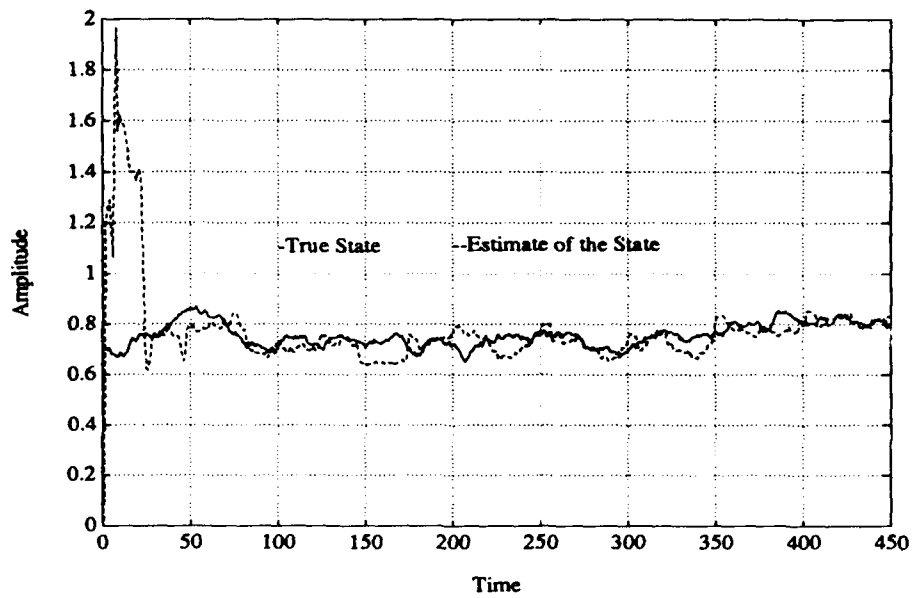


Figure 4. 4 True and the estimate value of the amplitude of the third harmonic

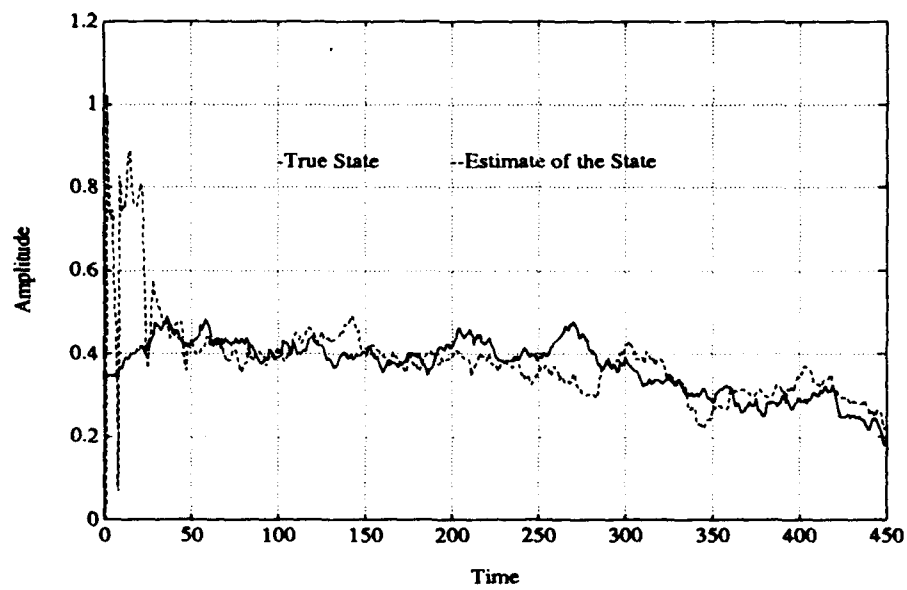


Figure 4. 5 True and the estimate value of the amplitude of the fourth harmonic

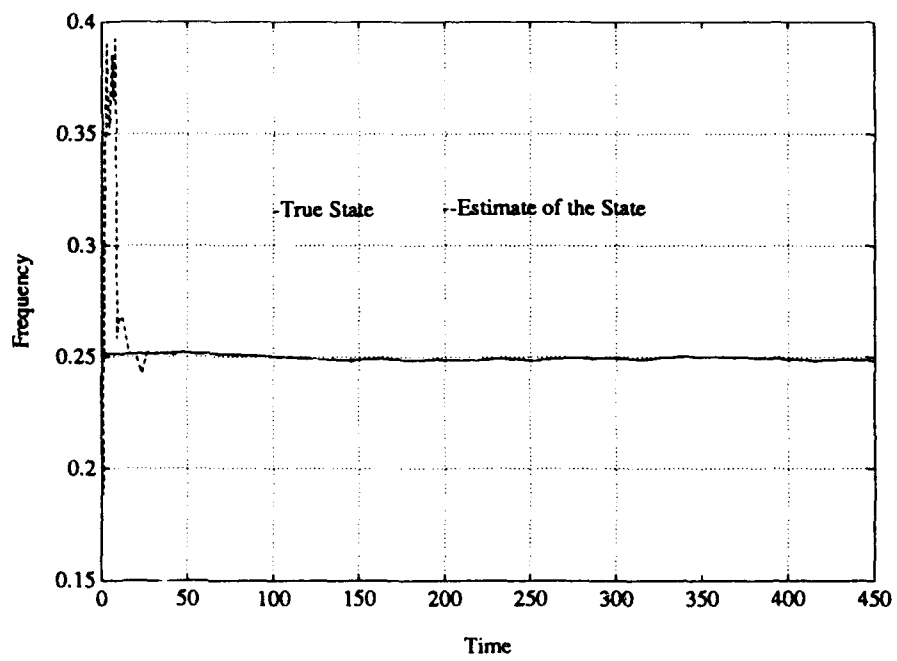


Figure 4. 6 True and the estimate value of the fundamental frequency

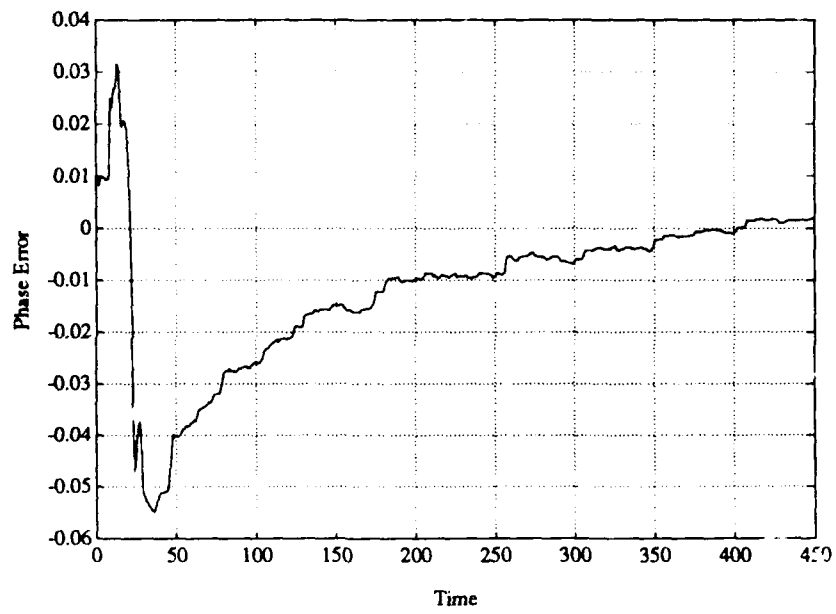


Figure 4.7 Phaseerror of the first harmonic

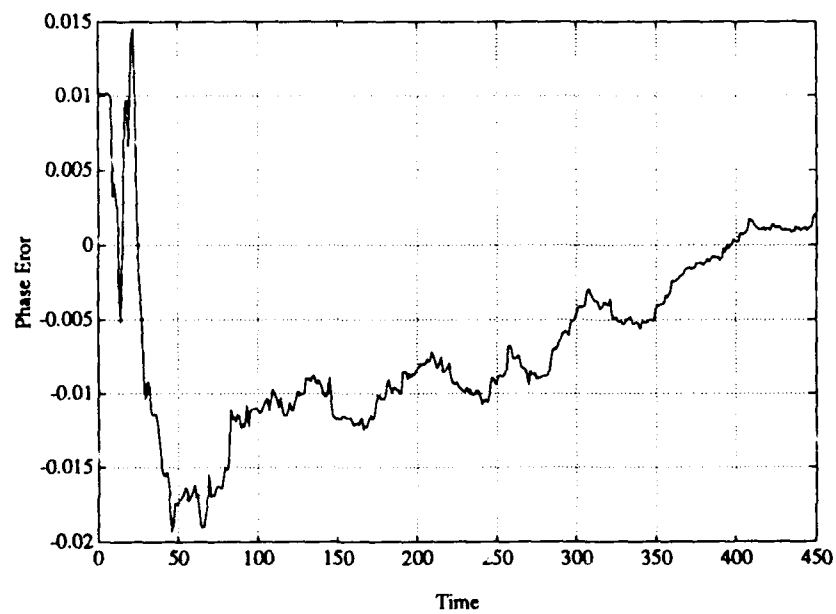


Figure 4.8 Phase error of the second harmonic

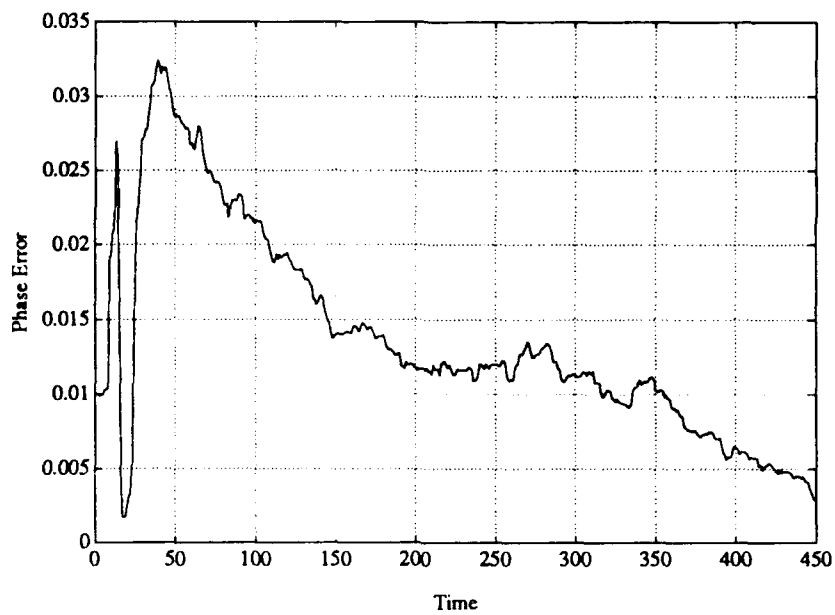


Figure 4. 9 Phase error of the third harmonic

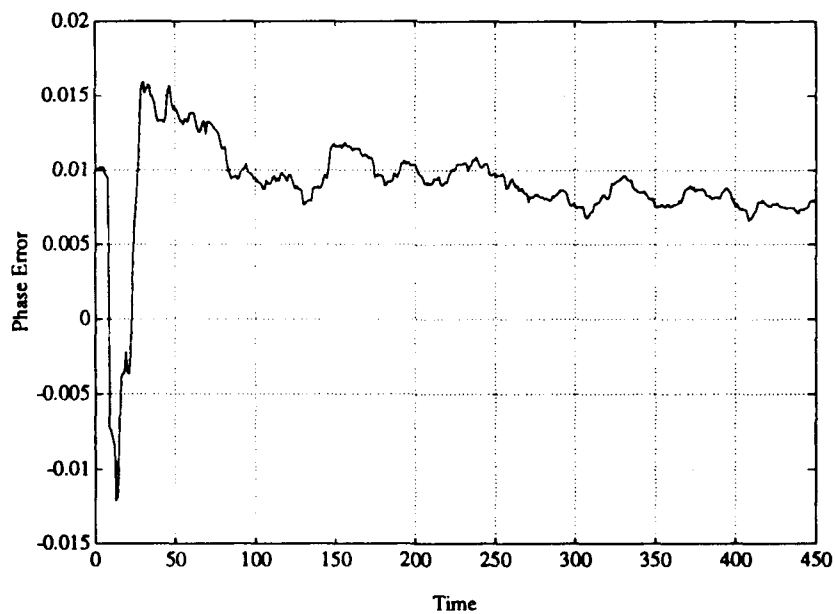


Figure 4. 10 Phase error of the fourth harmonic

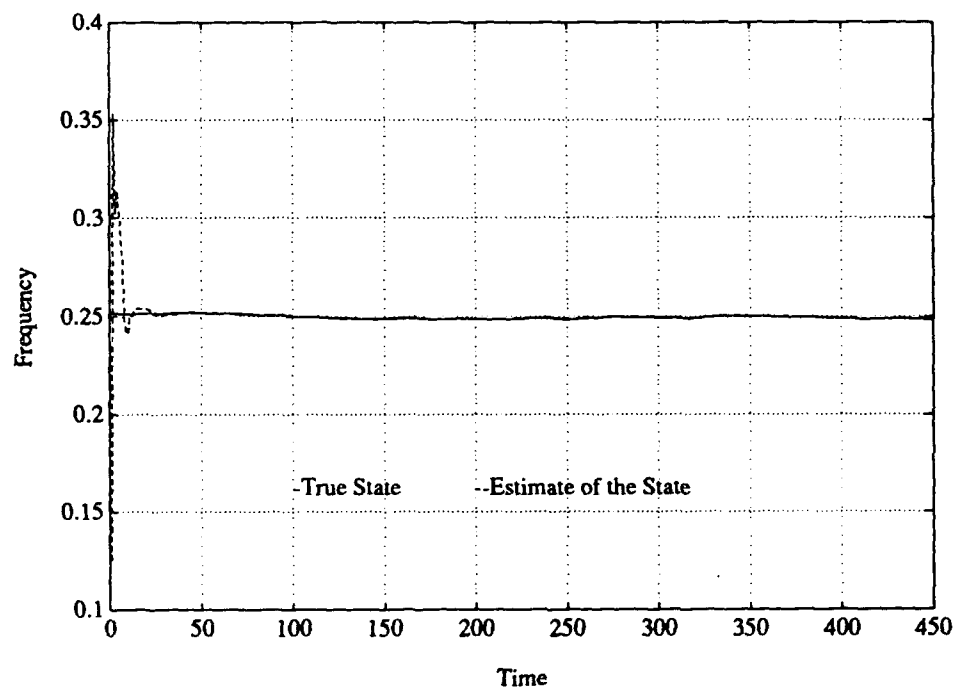


Figure 4. 13 Frequency tracking. (Two harmonic case)

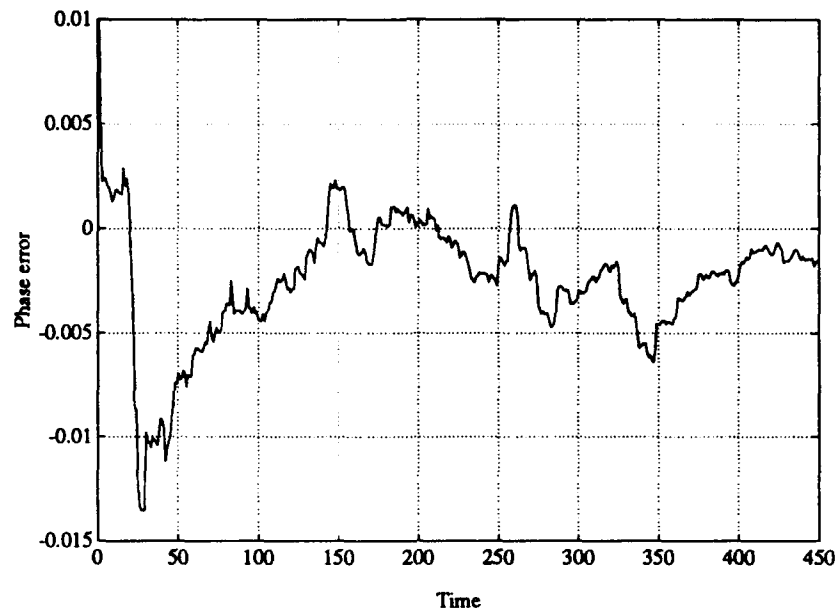


Figure 4.14 Phase error of the first harmonic. (Two harmonic case)

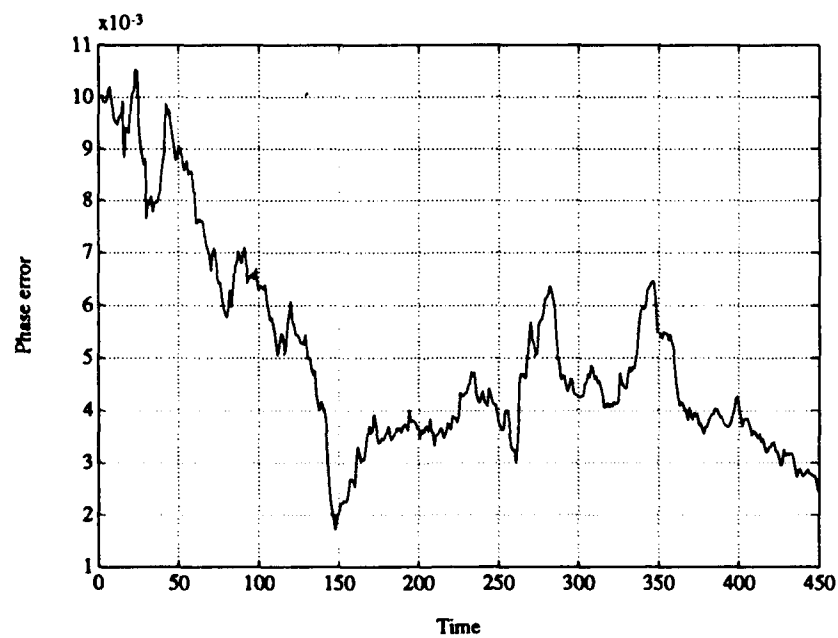


Figure 4.15 Phase error of the third harmonic. (Two harmonic case)

V. RECTANGULAR MODEL OF THE SYSTEM

This chapter applies the extended Kalman filter to the signal formulated in Chapter IV, but represents it in a different way. The signal is not periodic but has frequency, amplitude and phases that change slowly over time in a similar way to that in Chapter IV. Three sets of parameters (frequency, amplitude, and phase) define the signal. We will call the parameterization in this chapter the rectangular model as defined by Anderson and Parker [Ref. 7] and Anderson and James [Ref. 8]. In this representation $y(t)$ has amplitudes and phases with sine and cosine components, like

$$y(t) = a_1(t) \sin(\omega_f(t)t + \phi(t)) + \sum_{k=2}^{\infty} [a_k(t) \sin(k(\omega_f(t)t + \phi(t)))] + \sum_{k=2}^{\infty} [b_k(t) \cos(k(\omega_f(t)t + \phi(t)))] \quad (5.1)$$

The parameters of the system are $a_1(t)$, $a_2(t)$, ..., $a_m(t)$, $b_2(t)$, ..., $b_m(t)$, $\omega_f(t)$, and $\phi(t)$. The signal $y(t)$ has slow time varying frequency, amplitudes, and phases. That is:

$$\omega_f = \omega_f(t) \quad (5.2)$$

$$a_k = a_k(t) \quad (5.3)$$

$$b_k = b_k(t) \quad (5.4)$$

$$\phi = \phi(t) \quad (5.5)$$

where frequency, amplitude, and phase are nearly constant over several cycles.

$$\frac{|w_f(t+1) - w_f(t)|}{w_f(t)} \equiv 0 \quad (5.6)$$

$$\frac{|a_k(t+1) - a_k(t)|}{w_f(t)} \equiv 0 \quad (5.7)$$

$$\frac{|b_k(t+1) - b_k(t)|}{w_f(t)} \equiv 0 \quad (5.8)$$

$$\frac{|\phi(t+1) - \phi(t)|}{w_f(t)} \equiv 0 \quad (5.9)$$

In the representation of the signal, we did not include the $b_1(t)$ term cosinusoidal component of the fundamental frequency. This is because such a cosinusoidal frequency would give a phase shift in the fundamental, represented by a change in the phase of the fundamental.

Regardless of the working model, we will assume that the signal is contaminated by additive noise. This noise $v(t)$ will be white and Gaussian, which gives the measurement as:

$$z(t) = y(t) + v(t) \quad (5.10)$$

From these measurements, we will estimate $a_1(t)$, $a_2(t)$, ..., $a_m(t)$, $b_2(t)$, ..., $b_m(t)$, $w_f(t)$, $\phi(t)$. The estimation will be carried out up to the m th harmonic. The higher harmonics will be assumed to be insignificant for our purposes.

A. THE ESTIMATOR

The extended Kalman filter will be applied for the estimation of frequency, amplitude,

and phase of the measurement signal. In a similar way, we will denote the state-space model of the rectangular form. Therefore, Equations (4.9) and (4.11) remain unchanged, while the others become

$$x(t) = [a_1(t), a_2(t), \dots, a_m(t), b_2(t), \dots, b_m(t), \omega_f(t), \phi(t)]^T \quad (5.11)$$

$$\Phi = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{m-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.12)$$

$$h(x(t)) = a_1(t) \sin((\omega_f(t)t + \phi(t))) + \sum_{k=2}^m [a_k(t) \sin(k(\omega_f(t)t + \phi(t)))] + \sum_{k=2}^m [b_k(t) \cos(k(\omega_f(t)t + \phi(t)))] \quad (5.13)$$

The estimator Equations (4.18), (4.19), (4.20) and (4.21) of the polar form remain the same as for the rectangular form. The only difference appears in a Jacobian matrix as written below:

$$H(t) = \begin{bmatrix} \sin((\hat{\omega}_f(t|t-1)t + \hat{\phi}(t|t-1))) \\ \sin(2(\hat{\omega}_f(t|t-1)t + \hat{\phi}(t|t-1))) \\ \dots \\ \sin(m(\hat{\omega}_f(t|t-1)t + \hat{\phi}(t|t-1))) \\ \cos(2(\hat{\omega}_f(t|t-1)t + \hat{\phi}(t|t-1))) \\ \dots \\ \cos(m(\hat{\omega}_f(t|t-1)t + \hat{\phi}(t|t-1))) \\ \hat{\omega}_f(t|t-1) \\ \hat{\phi}(t|t-1) \end{bmatrix}^T \quad (5.14)$$

where,

$$\begin{aligned} \mathfrak{I}(t|t-1) = & \sum_{k=1}^m k \hat{a}_k(t|t-1) t \cos(k(\hat{w}_f(t|t-1)t + \hat{\phi}(t|t-1))) - \\ & \sum_{k=2}^m k \hat{b}_k(t|t-1) t \sin(k(\hat{w}_f(t|t-1)t + \hat{\phi}(t|t-1))) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{P}(t|t-1) = & \sum_{k=1}^m k \hat{a}_k(t|t-1) \cos(k(\hat{w}_f(t|t-1)t + \hat{\phi}(t|t-1))) - \\ & \sum_{k=2}^m k \hat{b}_k(t|t-1) \sin(k(\hat{w}_f(t|t-1)t + \hat{\phi}(t|t-1))). \end{aligned}$$

B. SIMULATIONS

Like the polar model, the rectangular model also has $2m+1$ states, with m representing the number of harmonics. We choose the number of harmonics $m=4$ to compare with the polar model. The measurements consisting of sine and cosine components are

$$\begin{aligned} z(t) = & a_1(t) \sin((2\pi 0.04t + 0.01)) + a_2(t) \sin(2(2\pi 0.04t + 0.01)) + \\ & a_3(t) \sin(3(2\pi 0.04t + 0.01)) + a_4(t) \sin(4(2\pi 0.04t + 0.01)) + \\ & b_2(t) \cos(2(2\pi 0.04t + 0.01)) + b_3(t) \cos(3(2\pi 0.04t + 0.01)) + \\ & b_4(t) \cos(4(2\pi 0.04t + 0.01)) + v(t). \end{aligned} \quad (5.15)$$

Here $v(t)$ is zero mean, white Gaussian noise with a variance of 0.1. The signal has

a fundamental frequency $\omega_f = 2\pi 0.04$ and amplitudes of a_k and b_k for the sine and the cosine components, respectively. They are defined as

$$a_k = \frac{a_{k-1}}{2} \quad k = 2, 3, 4 \quad (5.16)$$

$$b_k = \frac{a_{k-1}}{2} \quad k = 2, 3, 4 \quad (5.17)$$

The values of the amplitudes can be obtained from the desired signal to noise ratio. The program written in MATLAB (See Appendix C) is used for the simulations with the following initial conditions:

$$\hat{a}_1 = 2.4 \quad (5.18)$$

$$\hat{a}_2 = 0 \quad (5.19)$$

$$\hat{a}_3 = 0 \quad (5.20)$$

$$\hat{a}_4 = 0 \quad (5.21)$$

$$\hat{b}_2 = 1.45 \quad (5.22)$$

$$\hat{b}_3 = 0 \quad (5.23)$$

$$\hat{b}_4 = 0 \quad (5.24)$$

$$\hat{\omega}_f = 2\pi 0.03 \quad (5.25)$$

$$\hat{\phi} = 0 \quad (5.26)$$

$$P(0) = \text{diag}\{1, 0.5, 0.1, 0.04, 0.5, 0.1, 0.04, 0.8, 0.01\} \quad (5.27)$$

These initial values are chosen closely to those used in the polar form. The main difference appears in the initial value of the error covariance matrix. The program uses the same value of measurement and process noise as in the polar form.

Figure 5.1 shows the true and estimated values of the amplitude of the first harmonic. Tracking is achieved after $t = 50$. In Figures 5.2 and 5.3, the amplitudes of the sine component of the second and third harmonics show a similar response to the first one. Figure 5.4 shows the true and estimated values of the sine component of the fourth harmonic. The error looks worse than the others with the filter tracking after $t = 150$. Figures 5.5, 5.6 and 5.7 show the estimated cosine components. Their responses are similar to the sine components. Figure 5.8 shows the frequency tracking of the signal. The peak value of the estimation is large compared to the one estimated in the polar form. But, like the polar form, the rectangular model tracks the frequency after $t = 30$ and locks onto it very closely. Figure 5.9 shows the phase error of the signal. It has a steady state value of -0.0003 after $t = 70$.

We also applied a case with less than four harmonics to the rectangular model. The measurements are assumed to be

$$z(t) = a_1(t) \sin(2\pi 0.04t + 0.01) + a_3(t) \sin(3(2\pi 0.04t + 0.01)) + b_2(t) \cos(2(2\pi 0.04t + 0.01)) + v(t). \quad (5.28)$$

The second and the fourth harmonics of the sine component and the third and the fourth harmonics of the cosine component are assumed to be missing for the rectangular model. Similar to the polar form, we also rearranged the error covariance matrix for the rectangular form as

$$P(0) = \text{diag}\{1, 0.2, 0.1, 0.02, 1, 0.1, 0.01, 0.1, 0.001\}. \quad (5.29)$$

The MATLAB program used for the simulations is shown in Appendix D. Figure 5.10 shows the true and estimated values of the amplitude of the first harmonic. Its response is almost similar to the one estimated with four harmonics in the rectangular model. Figure

5.11 gives the estimation of the amplitude of the sine component of the third harmonic. Figure 5.12 shows the amplitude tracking of the cosine component of the second harmonic, which is very similar to the one obtained in the four harmonics case. The true and estimated values of the second and the fourth amplitude harmonics for the sine component were around zero. The same results apply to the third and the fourth amplitude harmonics of the cosine component. These graphs were not included. Figures 5.13 and Figure 5.14 show the frequency tracking and the phase error of the signal, respectively.

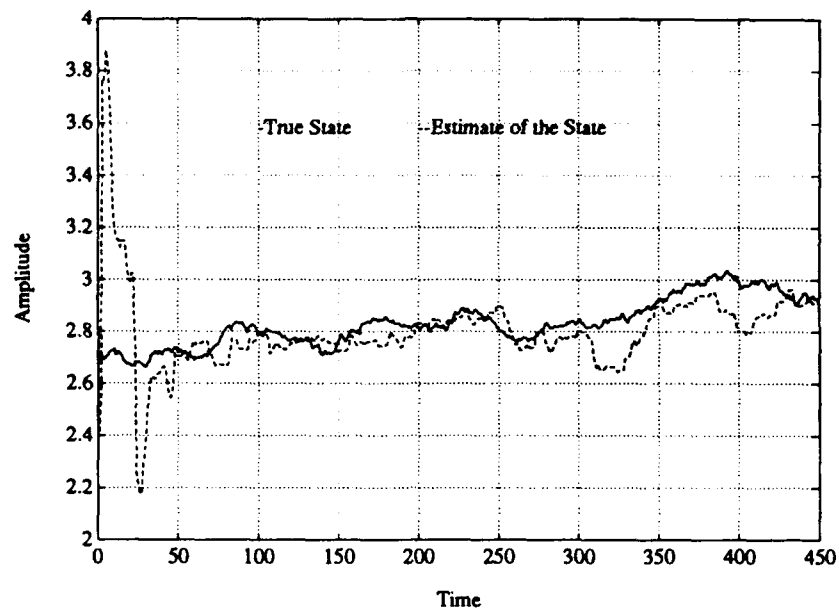


Figure 5.1 True and estimate value of the amplitude. First harmonic

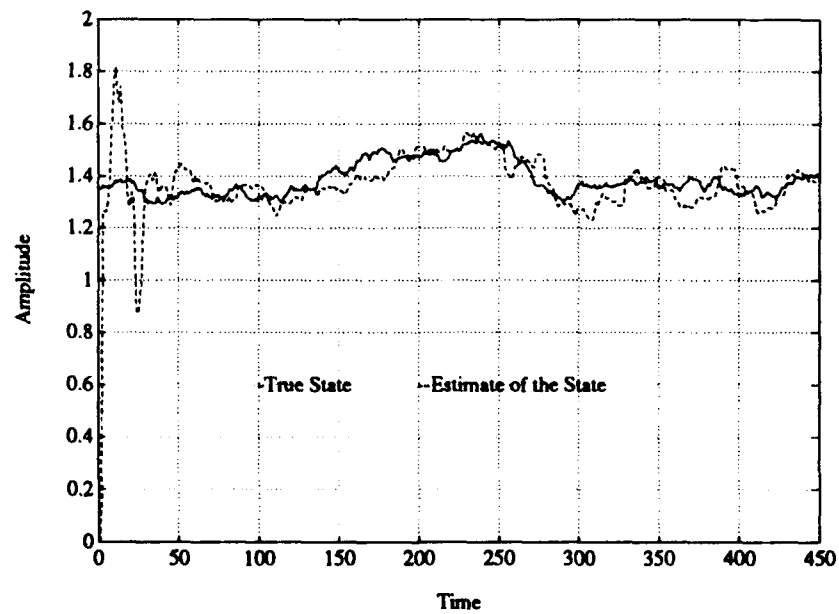


Figure 5.2 True and estimate value of the amplitude. Sine component, second harmonic



Figure 5.3 True and estimate value of the amplitude. Sine component, third harmonic

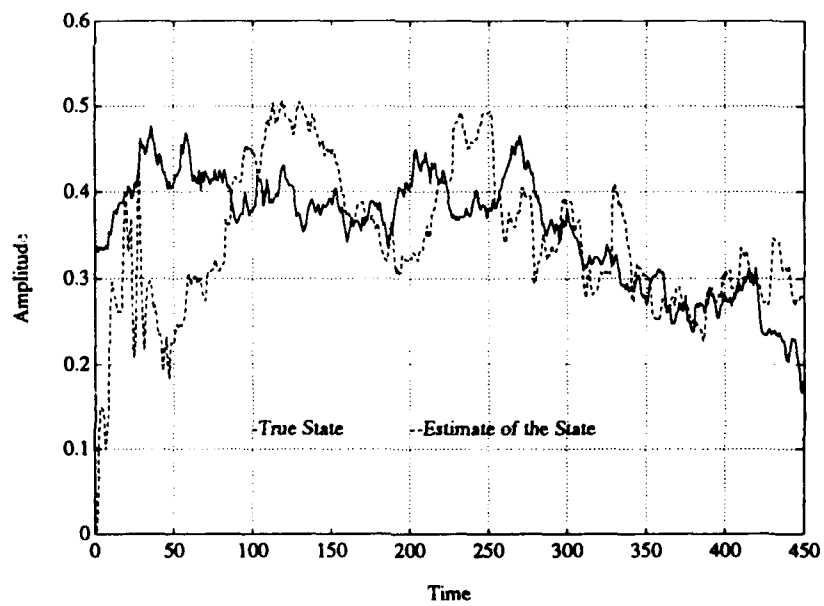


Figure 5.4 True and estimate value of the amplitude. Sine component, fourth harmonic

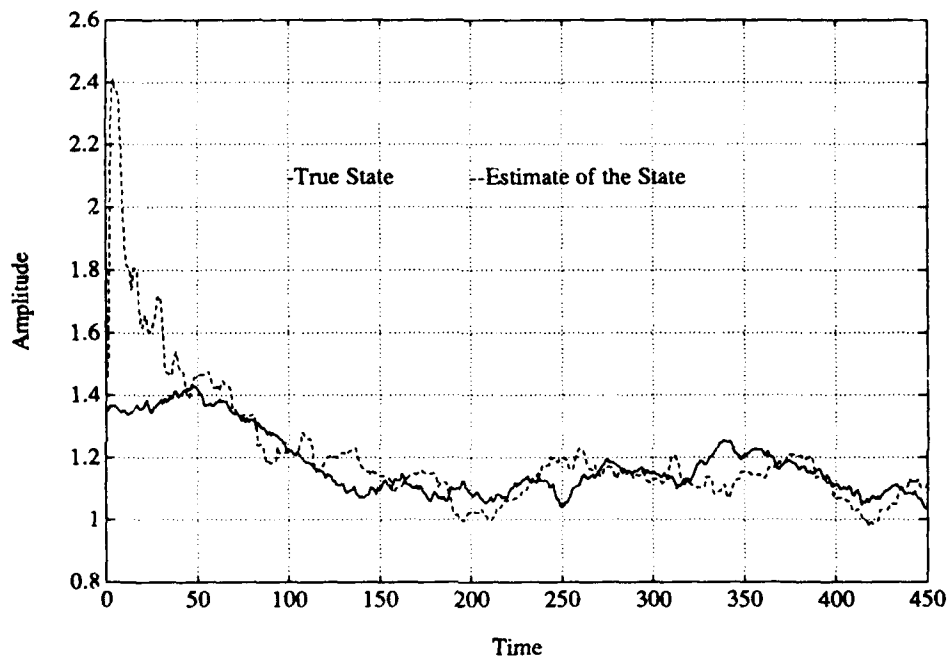


Figure 5.5 True and estimate value of the amplitude: Cosine component, second harmonic

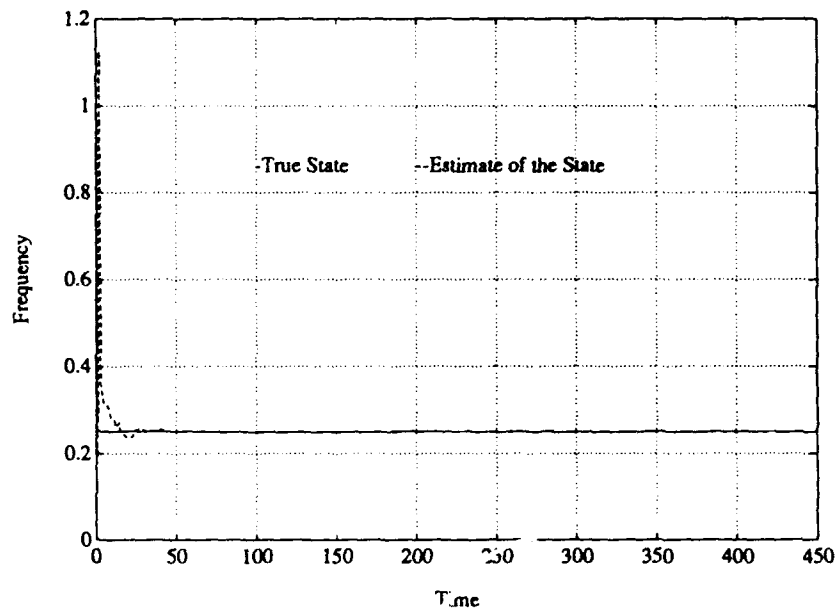


Figure 5.6 True and estimate value of the amplitude: Cosine component, third harmonic

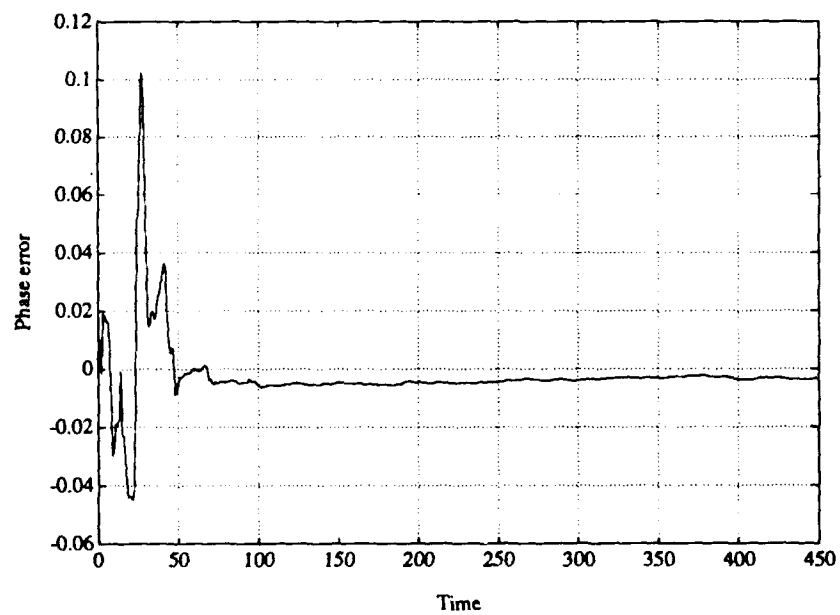


Figure 5.7 True and estimate value of the amplitude. Cosine component, fourth harmonic

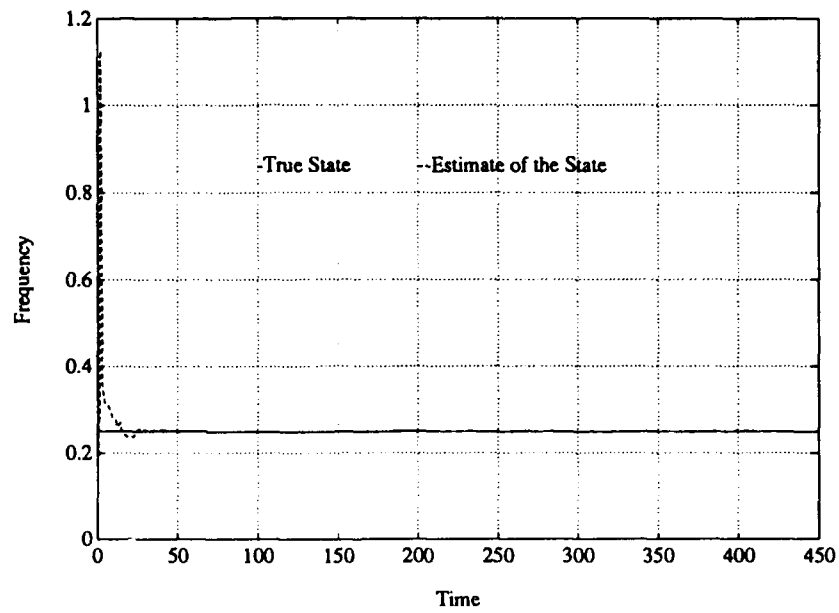


Figure 5.8 True and estimate value of the fundamental frequency

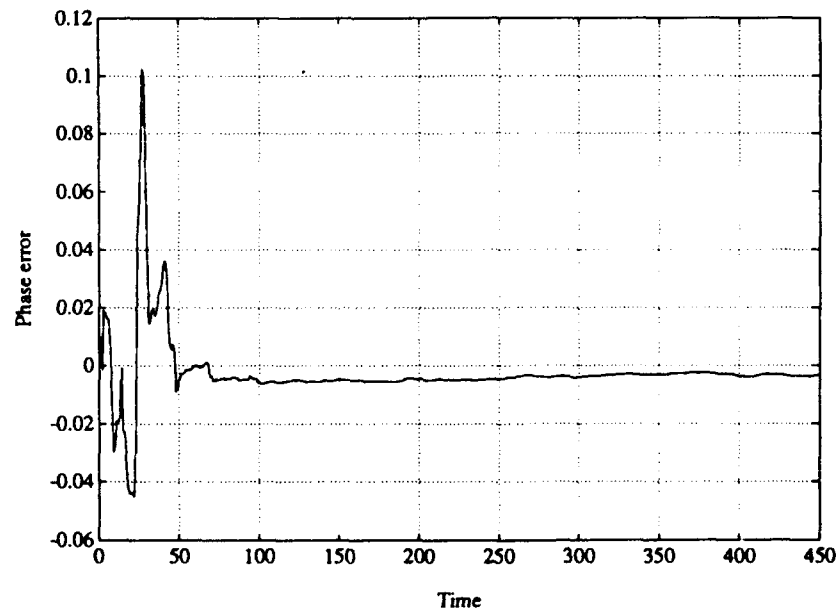


Figure 5.9 Phase error of the signal

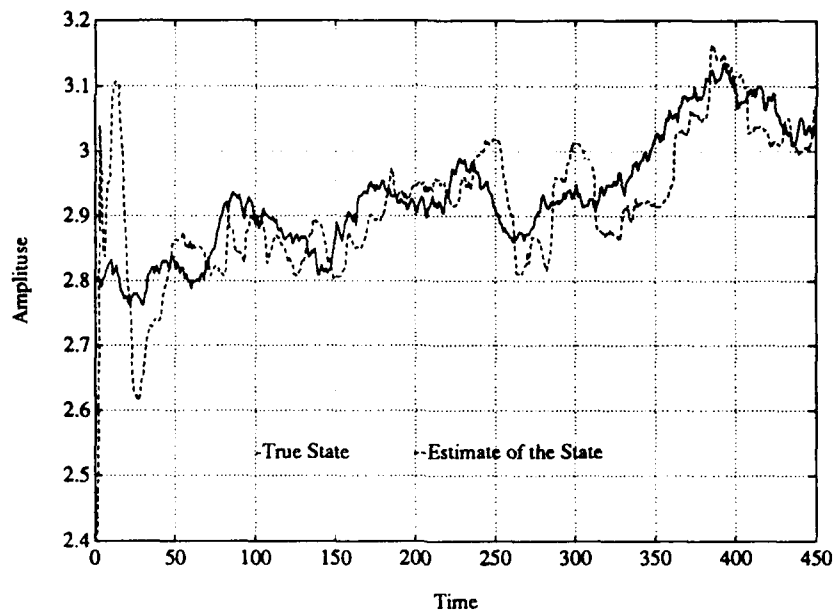


Figure 5.10 Amplitude tracking of the first harmonic. (Three harmonic case)

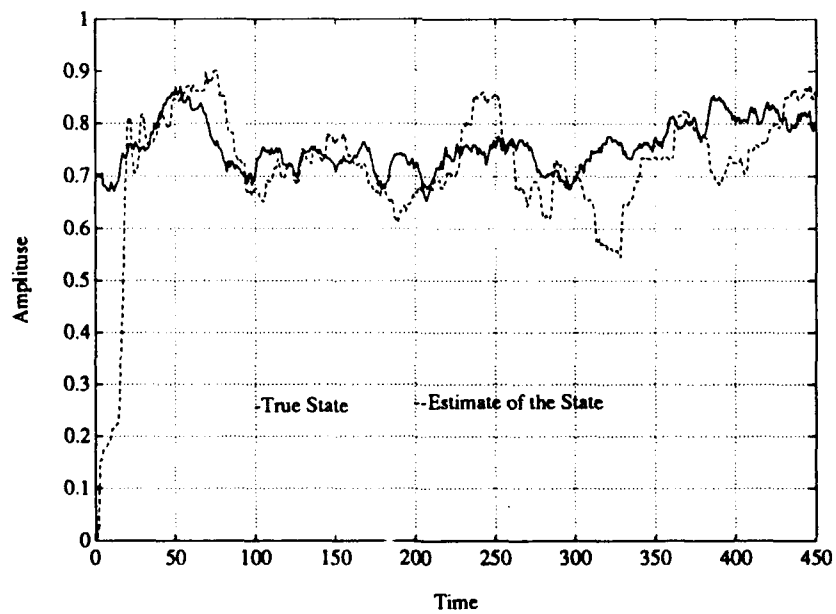


Figure 5.11 Amplitude tracking. Sine component, third harmonic. (Three harmonic case)

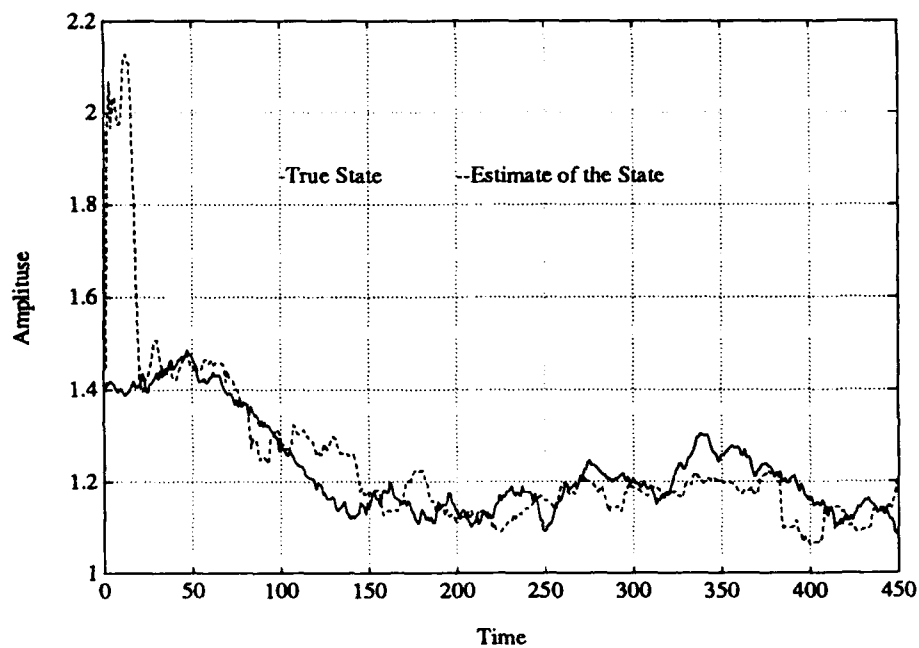


Figure 5.12 Amplitude tracking. Cosine component, second harmonic.
(Three harmonic case)

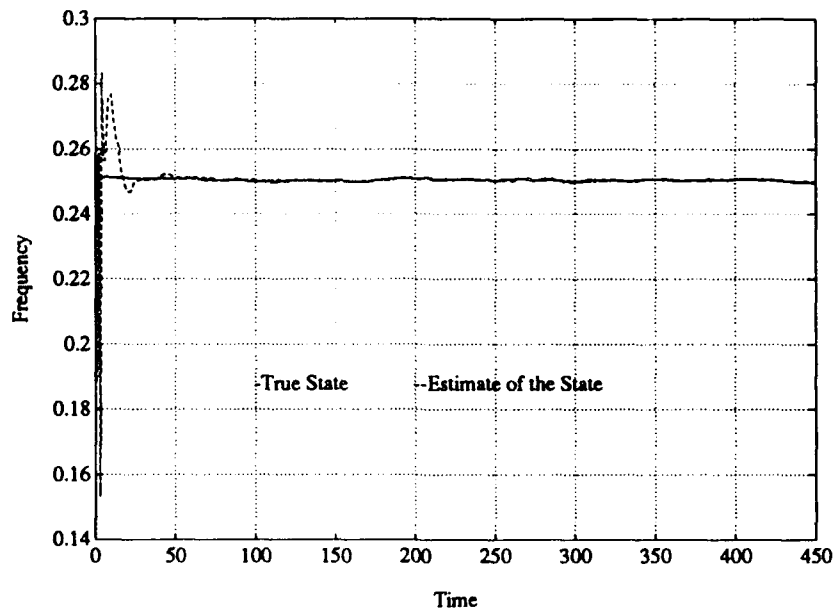


Figure 5.13 Frequency tracking. (Three harmonic case)

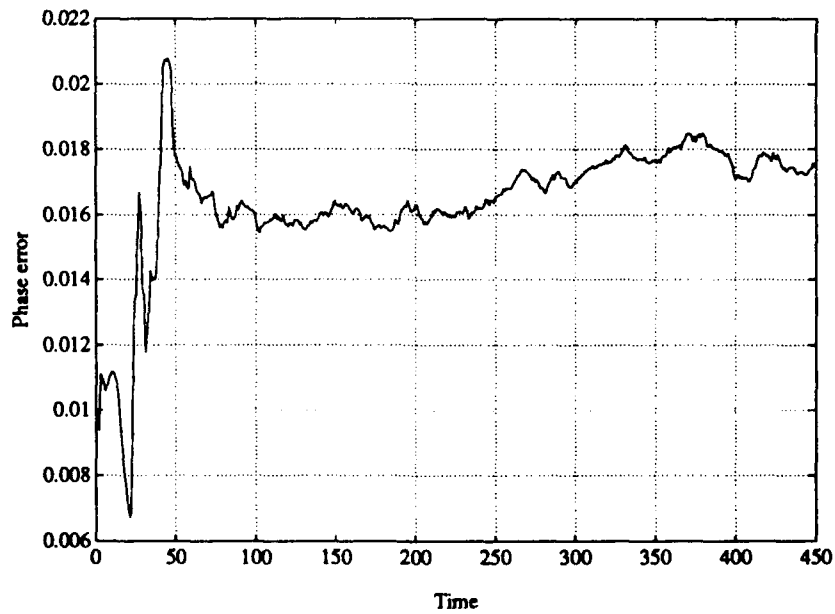


Figure 5.14 Phase error. (Three harmonic case)

VI. CONCLUSIONS

This thesis presents the estimation of frequency, amplitude, and phase of a signal using an extended Kalman filter algorithm. Both the polar and the rectangular model filter were implemented. In both models, the signal is contaminated by noise.

The simulations show that the performance of both filters are similar. Although there is little difference between the two models, the rectangular model seems to be able to estimate phase better than the polar model. For both filters, the tracking of the fundamental frequency plays a significant role. If the estimated value of the frequency locks onto a multiple of the true frequency, the filter cannot track the amplitudes. Each model offers a different advantage. The estimation of amplitudes of the signal is more convenient to measure in the polar model than in the rectangular model.

In all the simulations, an 18 dB signal-to-noise ratio was used. The performance of the two filters was poor when lower signal-to-noise ratios were used. The signals may have a Doppler shift. By adding another state, additional Doppler effects can be accounted for in future work.

APPENDIX A

%This program tracks the frequency, amplitude, and the phase of the nonperiodic signal in noise by using extended Kalman filter algorithm for the polar model.

% State space model

```
F=[1 0 0 0 0 0 0 0 0
    0 1 0 0 0 0 0 0 0
    0 0 1 0 0 0 0 0 0
    0 0 0 1 0 0 0 0 0
    0 0 0 0 1 0 0 0 0
    0 0 0 0 0 1 0 0 0
    0 0 0 0 0 0 1 0 0
    0 0 0 0 0 0 0 1 0
    0 0 0 0 0 0 0 0 1];
```

% Initial conditions of the true state

```
r1(1)=2.8; %Amplitude of the first component
r2(1)=r1(1)/2; %Amplitude of the second component
r3(1)=r2(1)/2; %Amplitude of the third component
r4(1)=r3(1)/2; %Amplitude of the fourth component
O1(1)=0.01; %Phase of the first component
O2(1)=0.01; %Phase of the second component
O3(1)=0.01; %Phase of the third component
O4(1)=0.01; %Phase of the fourth component
w(1)=2*pi*0.04; %Fundamental frequency
```

%Initial condition of the estimation

```
xkkm(:,1)=[2.6;0;0;0;2*pi*0.03;0;0;0;0];
```

R=0.1; % measurement noise

```

q1=1e-3; % process noise
q2=1e-7;
q3=1e-7;
Q=[q1 0 0 0 0 0 0 0 0
    0 q1 0 0 0 0 0 0 0
    0 0 q1 0 0 0 0 0 0
    0 0 0 q1 0 0 0 0 0
    0 0 0 0 q2 0 0 0 0
    0 0 0 0 0 q3 0 0 0
    0 0 0 0 0 0 q3 0 0
    0 0 0 0 0 0 0 q3 0
    0 0 0 0 0 0 0 0 q3];

%Error covariance matrix
P=[6 0 0 0 0 0 0 0 0
    0 5 0 0 0 0 0 0 0
    0 0 3 0 0 0 0 0 0
    0 0 0 2 0 0 0 0 0
    0 0 0 0 0.08 0 0 0 0
    0 0 0 0 0 0.001 0 0 0
    0 0 0 0 0 0 0.001 0 0
    0 0 0 0 0 0 0 0.001 0
    0 0 0 0 0 0 0 0 0.001];

%Jacobian matrix
H(1,:)=[0 0 0 0 0 0 0 0 0];

%Transpose of the Jacobian matrix
Ht(:,1)=H(1,:);

t=1:450; %Number of samples

%True state matrix
x(:,1)=[r1(1)
r2(1)
r3(1)
r4(1)
w(1)
O1(1)
O2(1)
O3(1)]

```

```

O4(1)];

for i = 1:450
rand('normal')

%Observation
z(i)=x(1,i)*sin(x(5,i)*i+x(6,i)) +
x(2,i)*sin(2*x(5,i)*i+x(7,i))+x(3,i)*sin(3*x(5,i)*i+x(8,i))+
x(4,i)*sin(4*x(5,i)*i+x(9,i))+0.1*rand(1);

he(i)=xkkm(1,i)*sin(xkkm(5,i)*i+xkkm(6,i))+xkkm(2,i)*sin(2*x
kkm(5,i)*i+xkkm(7,i))+xkkm(3,i)*sin(xkkm(5,i)*3*i+xkkm(8,i))
+xkkm(4,i)*sin(xkkm(5,i)*4*i+xkkm(9,i));

% Kalman gain
L(:,i)=P*H(i,:)'*inv(H(i,:)*P*H(i,:)' +R);

% Estimate update
xkk(:,i)=xkkm(:,i)+L(:,i)*(z(i) - he(i));
xkkm(:,i+1)=F*xkk(:,i);

%Tranpose of the Jacobian matrix
Ht(:,i+1)=[sin(xkkm(5,i)*i+xkkm(6,i));
sin(2*xkkm(5,i)*i+xkkm(7,i));
sin(xkkm(5,i)*3*i+xkkm(8,i));
sin(xkkm(5,i)*4*i+xkkm(9,i));

xkkm(1,i)*i*cos(xkkm(5,i)*i+xkkm(6,i))+xkkm(2,i)*2*i*cos(2*x
kkm(5,i)*i+xkkm(7,i))..
+xkkm(3,i)*3*i*cos(3*xkkm(5,i)*i+xkkm(8,i))+xkkm(4,i)*4*i*co
s(4*xkkm(5,i)*i+xkkm(9,i));
xkkm(1,i)*cos(xkkm(5,i)*i+xkkm(6,i));
xkkm(2,i)*cos(2*xkkm(5,i)*i+xkkm(7,i));
xkkm(3,i)*cos(3*xkkm(5,i)*i+xkkm(8,i));
xkkm(4,i)*cos(4*xkkm(5,i)*i+xkkm(9,i))];

%Jacobian
H(i+1,:)=Ht(:,i+1)';

% Error covariance update

```

```

P=F*(P-L(:,i)*H(i,:)*P)*F'+ Q;

v(:,i)=[rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-4
rand(1)*1e-4
rand(1)*1e-4
rand(1)*1e-4
rand(1)*1e-4];

%State update
x(:,i+1)=F*x(:,i) + v(:,i);
end
plot(t,x(1,1:i),t,xkk(1,1:i)),grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1a1

plot(t,x(2,1:i),t,xkk(2,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1a2

plot(t,x(3,1:i),t,xkk(3,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1a3

plot(t,x(4,1:i),t,xkk(4,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1a4

plot(t,x(5,1:i),t,xkk(5,1:i));pause,grid,xlabel('Time'),
ylabel('Frequency'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1a5

plot(t,x(6,1:i)-xkk(6,1:i));pause,grid,
xlabel('Time'),ylabel('Phase Error'),meta tez1a6

plot(t,x(7,1:i)-xkk(7,1:i));pause,grid,
xlabel('Time'),ylabel('Phase Error'),meta tez1a7

```



```
plot(t,x(8,1:i)-xkk(8,1:i));pause,grid,  
xlabel('Time'),ylabel('Phase Error'),meta tezla8
```

```
plot(t,x(9,1:i)-xkk(9,1:i));pause,grid,  
xlabel('Time'),ylabel('Phase Error'),meta tezla9
```

APPENDIX B

%This program computes the frequency, amplitude and the phase of the,nonperiodic signal in noise which consists of two harmonics for the polar model.It uses extended Kalman filter algorithm.

% State space model

```
F=[1 0 0 0 0 0 0 0 0
    0 1 0 0 0 0 0 0 0
    0 0 1 0 0 0 0 0 0
    0 0 0 1 0 0 0 0 0
    0 0 0 0 1 0 0 0 0
    0 0 0 0 0 1 0 0 0
    0 0 0 0 0 0 1 0 0
    0 0 0 0 0 0 0 1 0
    0 0 0 0 0 0 0 0 1];
```

% Initial conditions of the true state

```
r1(1)=2.8; %Amplitude of the first component
r2(1)=0;%r1(1)/2; %Amplitude of the second component
r3(1)=r1(1)/4; %Amplitude of the third component
r4(1)=0;%r3(1)/2; %Amplitude of the fourth component
O1(1)=0.01; %Phase of the first component
O2(1)=0;%0.01; %Phase of the second component
O3(1)=0.01; %Phase of the third component
O4(1)=0;%0.01; %Phase of the fourth component
w(1)=2*pi*0.04; %Fundamental frequency
```

%Initial condition of the estimation

```
xkkm(:,1)=[2.6;0;0;0;2*pi*0.02;0;0;0;0];
```

R=0.1; % measurement noise

```

q1=1e-3; % process noise
q2=1e-7;
q3=1e-7;
Q=[q1 0 0 0 0 0 0 0 0
    0 q1 0 0 0 0 0 0 0
    0 0 q1 0 0 0 0 0 0
    0 0 0 q1 0 0 0 0 0
    0 0 0 0 q2 0 0 0 0
    0 0 0 0 0 q3 0 0 0
    0 0 0 0 0 0 q3 0 0
    0 0 0 0 0 0 0 q3 0
    0 0 0 0 0 0 0 0 q3];

%Error covariance matrix
P=[2 0 0 0 0 0 0 0 0
    0 1 0 0 0 0 0 0 0
    0 0 0.5 0 0 0 0 0 0
    0 0 0 0.2 0 0 0 0 0
    0 0 0 0 0.04 0 0 0 0
    0 0 0 0 0 0.001 0 0 0
    0 0 0 0 0 0 0.001 0 0
    0 0 0 0 0 0 0 0.001 0
    0 0 0 0 0 0 0 0 0.001];

%Jacobian matrix
H(1,:)=[0 0 0 0 0 0 0 0 0];

%Transpose of the Jacobian matrix
Ht(:,1)=H(1,:);

t=1:450;

%True state matrix
x(:,1)=[r1(1)
r2(1)
r3(1)
r4(1)
w(1)
O1(1)
O2(1)

```

```

O3(1)
O4(1)];

for i = 1:450
rand('normal')

%Observation
z(i)=x(1,i)*sin(x(5,i)*i+x(6,i)) +
x(2,i)*sin(2*x(5,i)*i+x(7,i))+x(3,i)*sin(3*x(5,i)*i+x(8,i))+
x(4,i)*sin(4*x(5,i)*i+x(9,i))+0.1*rand(1);

he(i)=xkkm(1,i)*sin(xkkm(5,i)*i+xkkm(6,i))+xkkm(2,i)*sin(2*x
kkm(5,i)*i+xkkm(7,i))+xkkm(3,i)*sin(xkkm(5,i)*3*i+xkkm(8,i))
+xkkm(4,i)*sin(xkkm(5,i)*4*i+xkkm(9,i));

% Kalman gain
L(:,i)=P*H(i,:)'*inv(H(i,:)*P*H(i,:)+R);

% Estimate update
xkk(:,i)=xkkm(:,i)+L(:,i)*(z(i) - he(i));
xkkm(:,i+1)=F*xkk(:,i);

%Tranpose of the Jacobian matrix
Ht(:,i+1)=[sin(xkkm(5,i)*i+xkkm(6,i));
sin(2*xkkm(5,i)*i+xkkm(7,i));
sin(xkkm(5,i)*3*i+xkkm(8,i));
sin(xkkm(5,i)*4*i+xkkm(9,i));

xkkm(1,i)*i*cos(xkkm(5,i)*i+xkkm(6,i))+xkkm(2,i)*2*i*cos(2*x
kkm(5,i)*i+xkkm(7,i))..
+xkkm(3,i)*3*i*cos(3*xkkm(5,i)*i+xkkm(8,i))+xkkm(4,i)*4*i*co
s(4*xkkm(5,i)*i+xkkm(9,i));
xkkm(1,i)*cos(xkkm(5,i)*i+xkkm(6,i));
xkkm(2,i)*cos(2*xkkm(5,i)*i+xkkm(7,i));
xkkm(3,i)*cos(3*xkkm(5,i)*i+xkkm(8,i));
xkkm(4,i)*cos(4*xkkm(5,i)*i+xkkm(9,i))];

%Jacobian
H(i+1,:)=Ht(:,i+1)';

```

```

% Error covariance update
P=F*(P-L(:,i)*H(i,:)*P)*F'+ Q;

v(:,i)=[rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-4
rand(1)*1e-4
rand(1)*1e-4
rand(1)*1e-4
rand(1)*1e-4];

%State update
x(:,i+1)=F*x(:,i) + v(:,i);
end
plot(t,x(1,1:i),t,xkk(1,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1b1

plot(t,x(2,1:i),t,xkk(2,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1b2

plot(t,x(3,1:i),t,xkk(3,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1b3

plot(t,x(4,1:i),t,xkk(4,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1b4

plot(t,x(5,1:i),t,xkk(5,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1b5

plot(t,x(6,1:i)-xkk(6,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez1b6

plot(t,x(7,1:i)-xkk(7,1:i));pause,grid,xlabel('Time'),

```

```
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tezl7
```

```
plot(t,x(8,1:i)-xkk(8,1:i));pause,grid,xlabel('Time'),  
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tezl8
```

```
plot(t,x(9,1:i)-xkk(9,1:i));pause,grid,xlabel('Time'),  
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tezl9
```

```
%
```

APPENDIX C

%This program tracks the frequency, amplitude, and the phase of the nonperiodic signal in noise by using extended Kalman filter algorithm for the rectangular model.

% State space model

```
F=[1 0 0 0 0 0 0 0 0
    0 1 0 0 0 0 0 0 0
    0 0 1 0 0 0 0 0 0
    0 0 0 1 0 0 0 0 0
    0 0 0 0 1 0 0 0 0
    0 0 0 0 0 1 0 0 0
    0 0 0 0 0 0 1 0 0
    0 0 0 0 0 0 0 1 0
    0 0 0 0 0 0 0 0 1];
```

% Initial conditions of the true state

```
a(1)=2.7;%Amplitude of the first component.
a2(1)=a(1)/2;%Amplitude of the second component.
a3(1)=a(2)/2;%Amplitude of the third component.
a4(1)=a(3)/2;%Amplitude of the fourth component.
w(1)=2*pi*0.04;%Fundamental frequency.
O1(1)=0.01;%Phase of the first component
b2(1)=a(1)/2;%Phase of the second component
b3(1)=a(2)/2;%Phase of the third component
b4(1)=a(2)/2;%Phase of the fourth component
```

%Initial condition of the estimation

```
xkkm(:,1)=[2.4;0;0;0;1.450;0;0;2*pi*0.03;0];
```

R=0.1; % Measurement noise

q1=1e-3; % process noise

q2=1e-7;

q3=1e-7;

```

Q=[q1 0 0 0 0 0 0 0 0 0
    0 q1 0 0 0 0 0 0 0 0
    0 0 q1 0 0 0 0 0 0 0
    0 0 0 q1 0 0 0 0 0 0
    0 0 0 0 q1 0 0 0 0 0
    0 0 0 0 0 q1 0 0 0 0
    0 0 0 0 0 0 q1 0 0 0
    0 0 0 0 0 0 0 q2 0 0
    0 0 0 0 0 0 0 0 q3];

%Error covariance matrix.
Pkk=[1 0 0 0 0 0 0 0 0 0
      0 0.5 0 0 0 0 0 0 0 0
      0 0 0.1 0 0 0 0 0 0 0
      0 0 0 0.04 0 0 0 0 0 0
      0 0 0 0 0.5 0 0 0 0 0
      0 0 0 0 0 0.1 0 0 0 0
      0 0 0 0 0 0 0.04 0 0 0
      0 0 0 0 0 0 0 0.8 0 0
      0 0 0 0 0 0 0 0 0.001];

%Jacobian matrix.
H(1,:)= [0 0 0 0 0 0 0 0 0 0];

%Transpose of the Jacobian matrix.
Ht(:,1)=[0 0 0 0 0 0 0 0 0 0]';

t=1:450;%Number of samples

%True state matrix
x(:,1)=[a(1)
        a2(1)
        a3(1)
        a4(1)
        b2(1)
        b3(1)
        b4(1)
        w(1)
        O1(1)];

```



```

for i= 1:450
    rand('normal')

    z(i)=x(1,i)*sin(x(8,i)*i+x(9,i))+x(2,i)*sin(2*(x(8,i)*i+x(9,i)))+x(3,i)*sin(3*(x(8,i)*i+x(9,i)))+x(4,i)*sin(4*(i*x(8,i)+x(9,i)))+x(5,i)*cos(2*(x(8,i)*i+x(9,i)))+x(6,i)*cos(3*(x(8,i)*i+x(9,i)))+x(7,i)*cos(4*(x(8,i)*i+x(9,i)))+0.1*rand(1);

    %Variable
    je(i)=xkkm(1,i)*sin(xkkm(8,i)*i+xkkm(9,i))+xkkm(2,i)*sin(2*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(3,i)*sin(3*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(4,i)*sin(4*(i*xkkm(8,i)+xkkm(9,i)));

    %Variable
    ke(i)=xkkm(5,i)*cos(2*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(6,i)*cos(3*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(7,i)*cos(4*(xkkm(8,i)*i+xkkm(9,i)));

    he(i)=je(i)+ke(i);

    %Time update of the error covariance
    Pkkm=F*Pkk*F'+Q;

    % Kalman gain
    K(:,i)=Pkkm*H(i,:)'*inv(H(i,:)*Pkkm*H(i,:)' +R);

    % Estimate update
    xkk(:,i)=xkkm(:,i)+K(:,i)*(z(i) - he(i));
    xkkm(:,i+1)=F*xkk(:,i);

    %Variable
    c(i)=xkkm(1,i)*i*cos(xkkm(8,i)*i+xkkm(9,i))+xkkm(2,i)*2*i*cos(2*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(3,i)*3*i*cos(3*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(4,i)*4*i*cos(4*(i*xkkm(8,i)+xkkm(9,i)));

    %Variable
    d(i)=-xkkm(5,i)*2*i*sin(2*(xkkm(8,i)*i+xkkm(9,i)))-xkkm(6,i)*3*i*sin(3*(xkkm(8,i)*i+xkkm(9,i)))-xkkm(7,i)*4*i*sin(4*(xkkm(8,i)*i+xkkm(9,i)));

```

```

%Variable
f(i)=xkkm(1,i)*cos(xkkm(8,i)*i+xkkm(9,i))+xkkm(2,i)*2*cos(2*
(xkkm(8,i)*i+xkkm(9,i)))+xkkm(3,i)*3*cos(3*(xkkm(8,i)*i+xkkm
(9,i)))+xkkm(4,i)*4*cos(4*(i*xkkm(8,i)+xkkm(9,i)));
g(i)=-xkkm(5,i)*2*sin(2*(xkkm(8,i)*i+xkkm(9,i)))-
xkkm(6,i)*3*sin(3*(xkkm(8,i)*i+xkkm(9,i)))-
xkkm(7,i)*4*sin(4*(xkkm(8,i)*i+xkkm(9,i)));

%transpose of the Jacobian
Ht(:,i+1)=[sin(xkkm(8,i)*i+xkkm(9,i));
sin(2*(xkkm(8,i)*i+xkkm(9,i)));
sin(3*(xkkm(8,i)*i+xkkm(9,i)));
sin(4*(xkkm(8,i)*i+xkkm(9,i)));
cos(2*(xkkm(8,i)*i+xkkm(9,i)));
cos(3*(xkkm(8,i)*i+xkkm(9,i)));
cos(4*(xkkm(8,i)*i+xkkm(9,i)));
c(i)+d(i);
f(i)+g(i)];

%Jacobian
H(i+1,:)=Ht(:,i+1)';

%Measurement update of error covariance
Pkk=(eye(9) - K(:,i)*H'(i,:))*Pkkm;

v(:,i) = [rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-4
rand(1)*1e-4];

%State update
x(:,i+1)=F*x(:,i) + v(:,i);
end
plot(t,x(1,1:i),t,xkk(1,1:i)),grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez52c1

```

```
plot(t,x(2,1:i),t,xkk(2,1:i));pause,grid,xlabel('Time'),  
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tez52c2
```

```
plot(t,x(3,1:i),t,xkk(3,1:i));pause,grid,xlabel('Time'),  
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tez52c3
```

```
plot(t,x(4,1:i),t,xkk(4,1:i));pause,grid,xlabel('Time'),  
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tez52c4
```

```
plot(t,x(5,1:i),t,xkk(5,1:i));pause,grid,xlabel('Time'),  
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tez52c5
```

```
plot(t,x(6,1:i),t,xkk(6,1:i));pause,grid,xlabel('Time'),  
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tez52c6
```

```
plot(t,x(7,1:i),t,xkk(7,1:i));pause,grid,xlabel('Time'),  
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate  
of the State'),meta tez52c7
```

```
plot(t,x(8,1:i)-xkk(8,1:i));pause,grid,xlabel('Time'),  
ylabel('Phase Error'),gtext('--Estimate of the State'),meta  
tez52c8
```

```
plot(t,x(9,1:i),t,xkk(9,1:i));pause,grid,  
xlabel('Time'),ylabel('Phase Error'),meta tez1a9
```

APPENDIX D

%This program tracks the frequency, amplitude, and the phase of the nonperiodic signal in noise which consists of two harmonics. It uses extended Kalman filter algorithm for the rectangular model.

% State space model

```
F=[1 0 0 0 0 0 0 0 0 0
    0 1 0 0 0 0 0 0 0 0
    0 0 1 0 0 0 0 0 0 0
    0 0 0 1 0 0 0 0 0 0
    0 0 0 0 1 0 0 0 0 0
    0 0 0 0 0 1 0 0 0 0
    0 0 0 0 0 0 1 0 0 0
    0 0 0 0 0 0 0 1 0 0
    0 0 0 0 0 0 0 0 1 0
    0 0 0 0 0 0 0 0 0 1];
```

```
% Initial conditions of the true state
a(1)=2.7;%Amplitude of the first component.
a2(1)=0;%Amplitude of the second component.
a3(1)=a(2)/2;%Amplitude of the third component.
a4(1)=0;%Amplitude of the fourth component.
w(1)=2*pi*0.04;%Fundamental frequency.
O1(1)=0.01;%Phase of the first component
b2(1)=a(1)/2;%Phase of the second component
b3(1)=0;%Phase of the third component
b4(1)=0;%Phase of the fourth component
```

```
%Initial condition of the estimation
xkkm(:,1)=[2.4;0;0;0;1.450;0;0;2*pi*0.03;0];
```

```
R=0.1; % Measurement noise
```

```
q1=1e-3; % process noise
```

```

q2=1e-7;
q3=1e-7;
Q=[q1 0 0 0 0 0 0 0 0
    0 q1 0 0 0 0 0 0 0
    0 0 q1 0 0 0 0 0 0
    0 0 0 q1 0 0 0 0 0
    0 0 0 0 q1 0 0 0 0
    0 0 0 0 0 q1 0 0 0
    0 0 0 0 0 0 q1 0 0
    0 0 0 0 0 0 0 q2 0
    0 0 0 0 0 0 0 0 q3];

%Error covariance matrix.
Pkk=[1 0 0 0 0 0 0 0 0
      0 0.2 0 0 0 0 0 0 0
      0 0 0.1 0 0 0 0 0 0
      0 0 0 0.02 0 0 0 0 0
      0 0 0 0 1 0 0 0 0
      0 0 0 0 0 0.1 0 0 0
      0 0 0 0 0 0 0.01 0 0
      0 0 0 0 0 0 0 0.1 0
      0 0 0 0 0 0 0 0 0.001];

%Jacobian matrix.
H(1,:)= [0 0 0 0 0 0 0 0 0];

%Transpose of the Jacobian matrix.
Ht(:,1)=[0 0 0 0 0 0 0 0 0]';

t=1:450;%Number of samples

%True state matrix
x(:,1)=[a(1)
        a2(1)
        a3(1)
        a4(1)
        b2(1)
        b3(1)
        b4(1)
        w(1)
        O1(1)];

```

```

for i= 1:450
rand('normal')

```

```

z(i)=x(1,i)*sin(x(8,i)*i+x(9,i))+x(2,i)*sin(2*(x(8,i)*i+x(9,i)))+x(3,i)*sin(3*(x(8,i)*i+x(9,i)))+x(4,i)*sin(4*(i*x(8,i)+x(9,i)))+x(5,i)*cos(2*(x(8,i)*i+x(9,i)))+x(6,i)*cos(3*(x(8,i)*i+x(9,i)))+x(7,i)*cos(4*(x(8,i)*i+x(9,i)))+0.1*rand(1);

```

```

%Variable

```

```

je(i)=xkkm(1,i)*sin(xkkm(8,i)*i+xkkm(9,i))+xkkm(2,i)*sin(2*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(3,i)*sin(3*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(4,i)*sin(4*(i*xkkm(8,i)+xkkm(9,i)));

```

```

%Variable

```

```

ke(i)=xkkm(5,i)*cos(2*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(6,i)*cos(3*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(7,i)*cos(4*(xkkm(8,i)*i+xkkm(9,i)));

```

```

he(i)=je(i)+ke(i);

```

```

%Time update of the error covariance

```

```

Pkkm=F*Pkk*F'+Q;

```

```

% Kalman gain

```

```

K(:,i)=Pkkm*H(i,:)'*inv(H(i,:)*Pkkm*H(i,:)' +R);

```

```

% Estimate update

```

```

xkk(:,i)=xkkm(:,i)+K(:,i)*(z(i) - he(i));
xkkm(:,i+1)=F*xkk(:,i);

```

```

%Variable

```

```

c(i)=xkkm(1,i)*i*cos(xkkm(8,i)*i+xkkm(9,i))+xkkm(2,i)*2*i*cos(2*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(3,i)*3*i*cos(3*(xkkm(8,i)*i+xkkm(9,i)))+xkkm(4,i)*4*i*cos(4*(i*xkkm(8,i)+xkkm(9,i)));

```

```

%Variable

```

```

d(i)=-xkkm(5,i)*2*i*sin(2*(xkkm(8,i)*i+xkkm(9,i)))-
xkkm(6,i)*3*i*sin(3*(xkkm(8,i)*i+xkkm(9,i)))-
xkkm(7,i)*4*i*sin(4*(xkkm(8,i)*i+xkkm(9,i)));

```

```

%Variable

```

```

f(i)=xkkm(1,i)*cos(xkkm(8,i)*i+xkkm(9,i))+xkkm(2,i)*2*cos(2*
(xkkm(8,i)*i+xkkm(9,i)))+xkkm(3,i)*3*cos(3*(xkkm(8,i)*i+xkkm
(9,i)))+xkkm(4,i)*4*cos(4*(i*xkkm(8,i)+xkkm(9,i)));
g(i)=-xkkm(5,i)*2*sin(2*(xkkm(8,i)*i+xkkm(9,i)))-
xkkm(6,i)*3*sin(3*(xkkm(8,i)*i+xkkm(9,i)))-
xkkm(7,i)*4*sin(4*(xkkm(8,i)*i+xkkm(9,i)));

```

```

%transpose of the Jacobian

```

```

Ht(:,i+1)=[sin(xkkm(8,i)*i+xkkm(9,i));
sin(2*(xkkm(8,i)*i+xkkm(9,i)));
sin(3*(xkkm(8,i)*i+xkkm(9,i)));
sin(4*(xkkm(8,i)*i+xkkm(9,i)));
cos(2*(xkkm(8,i)*i+xkkm(9,i)));
cos(3*(xkkm(8,i)*i+xkkm(9,i)));
cos(4*(xkkm(8,i)*i+xkkm(9,i)));
c(i)+d(i);
f(i)+g(i)];

```

```

%Jacobian

```

```

H(i+1,:)=Ht(:,i+1)';

```

```

%Measurement update of error covariance

```

```

Pkk=(eye(9) - K(:,i)*H(i,:))*Pkkm;

```

```

v(:,i) = [rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-2
rand(1)*1e-4
rand(1)*1e-4];

```

```

%State update

```

```

x(:,i+1)=F*x(:,i) + v(:,i);

```

```

end
plot(t,x(1,1:i),t,xkk(1,1:i)),grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez52c1

plot(t,x(2,1:i),t,xkk(2,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez52c2

plot(t,x(3,1:i),t,xkk(3,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez52c3

plot(t,x(4,1:i),t,xkk(4,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez52c4

plot(t,x(5,1:i),t,xkk(5,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez52c5

plot(t,x(6,1:i),t,xkk(6,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez52c6

plot(t,x(7,1:i),t,xkk(7,1:i));pause,grid,xlabel('Time'),
ylabel('Amplitude'),gtext('-True State'),gtext('--Estimate
of the State'),meta tez52c7

plot(t,x(8,1:i)-xkk(8,1:i));pause,grid,xlabel('Time'),
ylabel('Frequency'),gtext('--Estimate of the State'),meta
tez52c8

plot(t,x(9,1:i),t,xkk(9,1:i));pause,grid,
xlabel('Time'),ylabel('Phase Error'),meta tez52c9

```


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